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PARAMETER-UNIFORM FITTED OPERATOR B-SPLINE COLLOCATION METHOD FOR SELF-ADJOINT SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract. In this paper, we develop a B-spline collocation method for the numerical solution of a self-adjoint singularly perturbed boundary value problem of the form

$$-\varepsilon(a(x)y')' + b(x)y(x) = f(x), \quad a(x) \geq a^* > 0,\ b(x) \geq b^* > 0,\ a'(x) \geq 0, \quad y(0) = \alpha, \quad y(1) = \beta.$$

We construct a fitting factor and use the B-spline collocation method, which leads to a tridiagonal linear system. The method is analyzed for parameter-uniform convergence. Several numerical examples are reported which demonstrate the efficiency of the proposed method.

Key words. B-spline collocation method, self-adjoint singularly perturbed boundary value problem, parameter-uniform convergence, boundary layer, fitted operator method

AMS subject classifications. 34D15, 30E25, 20B40

1. Introduction. We consider the following self-adjoint singularly perturbed boundary value problem,

$$Ly \equiv -\varepsilon(a(x)y')' + b(x)y(x) = f(x), \quad x \in \bar{\Omega} = [0, 1], \tag{1.1}$$

with the Dirichlet boundary conditions,

$$y(0) = \alpha, \quad y(1) = \beta, \tag{1.2}$$

where α and β are given constants and ε is a small positive parameter. The functions $a(x)$, $b(x)$, and $f(x)$ are sufficiently smooth and satisfy

$$a(x) \geq a^* > 0,\ b(x) \geq b^* > 0,\ a'(x) \geq 0. \tag{1.3}$$

Under these conditions the operator L admits the maximum principle [1].

These types of problems, in which a small parameter multiplies the highest derivative, are known as singular perturbation problems. They arise in the mathematical modeling of physical and chemical processes, for instance, reaction diffusion processes, chemical reactor theory, fluid mechanics, quantum mechanics, fluid dynamics, elasticity, etc. Schatz and Wahlbin [2] and Boglaev [3] solved this type of problem by using finite element techniques. Miller [4] gave a sufficient condition for the uniform first order convergence of a general three-point difference scheme, whereas Niijima [5] gave a uniformly second order accurate difference scheme. Kadalbajoo and Aggarwal [6] introduced the B-spline collocation method with a fitted mesh technique for self-adjoint singularly perturbed boundary value problems and proved second order uniform convergence of the method. It is well known that the solution of the SPP converges as $\varepsilon \to 0, \quad 0 \leq x \leq 1$, to the solution of the reduced problem obtained by putting $\varepsilon = 0$ in the original problem.

Parameter-uniform numerical methods [7, 8] are methods, whose numerical approximations U^N satisfy error bounds of the form

$$\|u_\varepsilon - U^N\| \leq C\vartheta(N), \quad \vartheta(N) \to 0 \text{ as } N \to \infty,$$

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where u_ε is the solution of the continuous problem, $\|.\|$ is the maximum pointwise norm, N is the number of mesh points (independent of ε) used, and C is a positive constant, which is independent of both ε and N. In other words, the numerical approximations U^N converge to u_ε for all values of ε in the range $0 < \varepsilon \ll 1$.

There are two well-known approaches to obtain small truncation error inside the boundary layer/layers when the perturbation parameter ε is very small. The first is a fitted method based on choosing a fine mesh in the boundary layer/layers region [8, 9, 10, 11] and the second is based on fitted operator methods, i.e., a difference formula reflecting the behavior of the solution in the boundary layer/layers. In this paper, we use the second strategy, and the proposed method, based on a suitably designed fitted operator to the interior layer, is shown to converge with $\vartheta(N) = 1/N^2$.

2. Continuous problem. The bounds for the solutions and its derivatives are given in this section. Furthermore, the bounds for the smooth and singular components and their derivatives are also given. We first give the maximum principle and stability estimates for the solution of the problem (1.1) and (1.2).

LEMMA 2.1 (Continuous maximum principle). *Let $\phi \in C^2(\bar{\Omega})$, satisfying $\phi(0) \geq 0$, $\phi(1) \geq 0$ and $L\phi(x) \geq 0 \; \forall \, x \in \Omega$. Then $\phi(x) \geq 0 \; \forall \, x \in \bar{\Omega}$.*

Proof. The proof is by contradiction. Suppose that there is a point $x^* \in \bar{\Omega}$, such that $\phi(x^*) < 0$ and $\phi(x^*) = \min\limits_{0 \leq x \leq 1} \phi(x)$. It is clear from the given conditions that $x^* \notin \{0, 1\}$. Therefore $\phi'(x^*) = 0$ and $\phi''(x^*) \geq 0$. Thus, we have

$$\begin{aligned} L\phi(x) \mid_{x=x^*} &= -\varepsilon(a(x)\phi'(x))' + b(x)\phi(x) \mid_{x=x^*} \\ &= -\varepsilon a(x)\phi''(x) - \varepsilon a'(x)\phi'(x) + b(x)\phi(x) \mid_{x=x^*} < 0, \end{aligned}$$

which is a contradiction. It follows that $\phi(x^*) \geq 0$ and so $\phi(x) \geq 0 \; \forall \, x \in \bar{\Omega}$. ∎

LEMMA 2.2 (Stability). *Consider the problem (1.1) and (1.2). If $\phi(x)$ is the solution of this problem, then for some positive constant C, we have*

$$\|\phi(x)\| \leq C \left(\max(\mid \alpha \mid, \mid \beta \mid) + \frac{1}{b^*}\|f\| \right), \quad \forall \, x \in \bar{\Omega}.$$

Proof. Consider the barrier functions

$$\Psi^{\pm}(x) = C \left(\max(\mid \alpha \mid, \mid \beta \mid) + \frac{1}{b^*}\|f\| \right) \pm \phi(x).$$

Then it can be easily seen that $\Psi^{\pm}(0) \geq 0$, $\Psi^{\pm}(1) \geq 0$ and for all $x \in \Omega$, $L\Psi^{\pm}(x) \geq 0$, by a proper choice of C. Therefore, by applying Lemma 2.1, we obtain $\Psi^{\pm}(x) \geq 0 \; \forall \, x \in \bar{\Omega}$, which gives the required estimate. ∎

3. Reduction to normal form. Rewrite (1.1) as

$$-\varepsilon a(x)y'' - \varepsilon a'(x)y' + b(x)y = f(x),$$

or

$$y'' + P(x)y' + Q(x)y = F(x), \tag{3.1}$$

where

$$P(x) = \frac{a'(x)}{a(x)}, \quad Q(x) = -\frac{b(x)}{\varepsilon a(x)}, \quad \text{and} \quad F(x) = -\frac{f(x)}{\varepsilon a(x)}.$$

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Let

$$U(x) = \exp\left(-\frac{1}{2}\int_0^x P(\zeta)\,d\zeta\right).$$

Consider the transformation

$$y(x) = U(x)V(x). \tag{3.2}$$

Then (3.1) can be written in the normal form as

$$V'' + \tilde{W}(x)V = \tilde{Z}(x), \tag{3.3}$$

with

$$V(0) = \frac{y(0)}{U(0)} = \eta_1, \quad V(1) = \frac{y(1)}{U(1)} = \eta_2, \quad \eta_1, \eta_2 \in R,$$

where

$$\tilde{W}(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2, \quad \tilde{Z}(x) = F(x)\exp\left(\frac{1}{2}\int_0^x P(\zeta)\,d\zeta\right).$$

Multiplying (3.3) throughout by $-\varepsilon$, we get

$$\tilde{L}V \equiv -\varepsilon V'' + W(x)V = Z(x), \tag{3.4}$$

with

$$V(0) = \eta_1, \quad V(1) = \eta_2, \tag{3.5}$$

where

$$W(x) = -\varepsilon\tilde{W}(x), \quad Z(x) = -\varepsilon\tilde{Z}(x), \quad \text{and} \quad W(x) \geq W^* > 0.$$

Note that

$$\begin{aligned} Z(x) &= -\varepsilon\tilde{Z}(x) \\ &= -\varepsilon F(x)\exp\left(\frac{1}{2}\int_0^x P(\zeta)\,d\zeta\right) \\ &= \frac{f(x)}{a(x)}\exp\left(\frac{1}{2}\int_0^x P(\zeta)\,d\zeta\right). \end{aligned}$$

This shows that $Z(x)$ is independent of ε. However, $W(x)$ may or may not depend on ε.

Roos [12] constructed a global uniformly convergent (in ε) scheme for (3.4) and (3.5), by replacing the coefficients by piecewise polynomials and solved the resulting problem exactly. To solve (3.4) and (3.5), Surla and Jerković [13] derived a difference scheme via a spline in tension and obtained error estimates of the form $O(h, \min(h, \varepsilon))$. O'Riordan and Stynes [14, 15, 16, 17] introduced the concept of freezing the coefficients by considering the piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the coefficient terms $W(x)$ and $Z(x)$ of the singularly perturbed boundary value problem (3.4) and (3.5). Stojanovic [18] gave an optimal difference scheme by considering the quadratic interpolating splines instead of piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the coefficient $Z(x)$. We define the fitting problem associated with (3.4) and (3.5) by

$$L_\sigma \equiv -\sigma(x, \varepsilon)V'' + W(x)V = Z(x), \tag{3.6}$$

with

$$V(0) = \eta_1, \quad V(1) = \eta_2, \tag{3.7}$$

where $\sigma(x, \varepsilon)$ is a fitting factor, which is to be determined subsequently.

4. B-Spline collocation method. In this section, we describe a B-spline collocation method to obtain the approximate solution of boundary value problems (3.6) and (3.7). Let $P \equiv \{0 = x_0 < x_1 < x_2 \ldots < x_{N-1} < x_N = 1\}$ be the partition of $\bar{\Omega}$ with uniform spacing $h = 1/N$. We include two more points on each side of the partition P as $x_{-2} < x_{-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$. Then $P \equiv \{x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2 \ldots < x_{N-1} < x_N = 1 < x_{N+1} < x_{N+2}\}$. Let $L_2(\bar{\Omega})$ be the space of all square integrable functions on $\bar{\Omega}$ and let X be a linear subspace of $L_2(\bar{\Omega})$. We use the cubic B-spline basis functions $B_i(x)$ for $i = 0(1)N$; see [19].

It is easy to see that each $B_i(x)$ is also a piecewise cubic with knots at $\bar{\Omega}_N$ and $B_i(x) \in X$. Thus, $B_i(x)$ is twice continuously differentiable $\forall\ x \in R$. The values of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at the nodal points x_i's are given in Table 4.1.

TABLE 4.1
B-Spline basis values.

	Nodal values				
	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i'(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$B_i''(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

Let $\mathbf{B} = \{B_{-1}, B_0, B_1, ..., B_{N+1}\}$ and let $\Phi_3(\bar{\Omega}_N) = \left\{\phi_i : \phi_i = \sum_{i=-1}^{N+1} k_i B_i, \quad k_i \in R\right\}$. The functions $B_i(x)$ are linearly independent on $\bar{\Omega}$. Thus, $\Phi_3(\bar{\Omega}_N)$ is $(N+3)$-dimensional subspace of X. Let L_σ be a linear operator with domain X and with range in X. We seek a function $S(x) \in \Phi_3(\bar{\Omega}_N)$ that approximates the solution of boundary value problem (3.6) and (3.7), represented as

$$S(x) = \sum_{i=-1}^{N+1} c_i B_i(x), \tag{4.1}$$

where c_i are unknown real coefficients. Here we have introduced two extra cubic B-splines, B_{-1} and B_{N+1} to satisfy the boundary conditions. Therefore, we have

$$L_\sigma S(x_i) = Z(x_i), \qquad 0 \le i \le N, \tag{4.2}$$

and

$$S(x_0) = \eta_1, \qquad S(x_N) = \eta_2. \tag{4.3}$$

Using Table 4.1 and equation (4.1), the system of collocation (4.2), together with boundary conditions (4.3), gives a system of $(N+1)$ linear equations in $(N+3)$ unknowns,

$$(-6\sigma_i/h^2 + W_i)c_{i-1} + (12\sigma_i/h^2 + 4W_i)c_i + (-6\sigma_i/h^2 + W_i)c_{i+1} = Z_i, \tag{4.4}$$

for $0 \le i \le N$, where $W(x_i) = W_i$, $Z(x_i) = Z_i$ and σ_i is a fitting factor which is to be determined. The given boundary conditions become

$$c_{-1} + 4c_0 + c_1 = \eta_1 \tag{4.5}$$

and

$$c_{N-1} + 4c_N + c_{N+1} = \eta_2. \tag{4.6}$$

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Thus, (4.4), (4.5), and (4.6) lead to a $(N+3)\times(N+3)$ system with $(N+3)$ unknowns $c^N = (c_{-1}, c_0, c_1, ..., c_{N+1})^t$. Now eliminating c_{-1} from the first equation of (4.4) and from (4.5), we get

$$(36\sigma_0/h^2)c_0 = Z_0 - \eta_1(-6\sigma_0/h^2 + W_0). \tag{4.7}$$

Similarly, eliminating c_{N+1} from the last equation of (4.4) and Eq. (4.6), we obtain

$$(36\sigma_N/h^2)c_N = Z_N - \eta_2(-6\sigma_N/h^2 + W_N). \tag{4.8}$$

Taking (4.7) and (4.8) with the second through $(N-1)$st equations of (4.4), we are lead to a system of $(N+1)$ linear equations,

$$Tc^N = d^N, \tag{4.9}$$

in $(N+1)$ unknowns $c^N = (c_0, c_1, \dots, c_N)^t$ with right hand side

$$d^N = (Z_0 - \eta_1(-6\sigma_0/h^2 + W_0), Z_1, \dots, Z_{N-1}, Z_N - \eta_2(-6\sigma_N/h^2 + W_N)).$$

The coefficient matrix T is given by

$$\begin{bmatrix} \frac{36\sigma_0}{h^2} & 0 & 0 & 0 & \dots & 0 \\ \frac{-6\sigma_1}{h^2}+W_1 & \frac{12\sigma_1}{h^2}+4W_1 & \frac{-6\sigma_1}{h^2}+W_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \frac{-6\sigma_i}{h^2}+W_i & \frac{12\sigma_i}{h^2}+4W_i & \frac{-6\sigma_i}{h^2}+W_i & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \frac{-6\sigma_{N-1}}{h^2}+W_{N-1} & \frac{12\sigma_{N-1}}{h^2}+4W_{N-1} & \frac{-6\sigma_{N-1}}{h^2}+W_{N-1} \\ 0 & 0 & 0 & 0 & 0 & \frac{36\sigma_N}{h^2} \end{bmatrix}.$$

Since $W(x) > 0$, it can be easily seen that the matrix T is strictly diagonally dominant and hence nonsingular. Since T is nonsingular, we can solve the system $Tc^N = d^N$ for $c_0, c_1, \dots, c_N$ and substitute into the boundary condition (4.5) and (4.6) to obtain c_{-1} and c_{N+1}. Hence the method of collocation using a basis of cubic B-splines applied to problem (3.6) and (3.7) has a unique solution $S(x)$ given by (4.1).

5. Determination of a fitting factor. In order to obtain a fitting factor, we shall use the following lemma [20].

LEMMA 5.1. *Let $V(x) \in C^4(\bar{\Omega})$, and let $W'(0) = W'(1) = 0$, then the solution of the problem (3.4) and (3.5) is of the form*

$$V(x) = u(x) + v(x) + w(x),$$

where

$$u(x) = p\exp\left(-x\sqrt{W(0)/\varepsilon}\right), \quad v(x) = q\exp\left(-(1-x)\sqrt{W(1)/\varepsilon}\right),$$

where p and q are bounded functions of ε independent of x and

$$\mid w^{(j)}(x) \mid \leq C(1+\varepsilon^{(1-j)/2}), \quad j = 0(1)4,$$

C is a constant independent of ε.

It is clear that the matrix T is inverse monotone if $h^2W_j/6\sigma_j \leq 1$, thus taking the fitting factor of the form,

$$\sigma_j = \frac{h^2 W_j}{6}\mu(\rho_j),$$

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where $\rho_j = \sqrt{W_j/\varepsilon}$ and $\mu(\rho_j)$ is to be determined. We want to find a fitting factor such that the truncation error for the boundary layer function is equal to zero when $W(x)$ is constant.

From the condition $Tu_i = 0$, for $W(x) = W =$ constant, we have

$$(-6\sigma/h^2 + W)u_{i-1} + (12\sigma/h^2 + 4W)u_i + (-6\sigma/h^2 + W)u_{i+1} = 0, \tag{5.1}$$

for $0 \le i \le N$, where

$$u_{i-1} = p\exp\left(-x_{i-1}\sqrt{W(0)/\varepsilon}\right) = u_i \exp\left(h\sqrt{W(0)/\varepsilon}\right),$$
$$u_{i+1} = p\exp\left(-x_{i+1}\sqrt{W(0)/\varepsilon}\right) = u_i \exp\left(-h\sqrt{W(0)/\varepsilon}\right).$$

Substituting in (5.1), a simple calculation gives

$$\sigma(\rho) = \frac{h^2 W}{6}\mu(\rho), \quad \text{when } W(x) = \text{constant}, \tag{5.2}$$

$$\sigma(\rho_j) = \frac{h^2 W_j}{6}\mu(\rho_j), \quad \text{when } W(x) \neq \text{constant}, \tag{5.3}$$

where

$$\mu(\rho_j) = \frac{1 + 2\cosh^2(\rho_j h/2)}{2\sinh^2(\rho_j h/2)}.$$

6. Convergence of the scheme.

In order to prove the uniform convergence of the scheme, first we shall prove the following lemma.

LEMMA 6.1. *Let σ_j be the fitting factor determined in Section 5, then σ_j approximates ε with an error $O(h^2)$, i.e.,*

$$|\ \sigma_j - \varepsilon\ | \le Ch^2,$$

for some positive constant C.

Proof. It is clear from (5.3) that

$$0 \le \sigma_j \le Ch^2.$$

There are two cases:
Case I. $\varepsilon \le h^2$. In this case we have

$$|\ \sigma_j - \varepsilon\ | \le |\ \sigma_j\ | + |\ \varepsilon\ | \le Ch^2. \tag{6.1}$$

Case II. $h^2 \le \varepsilon$. In this case we have

$$\begin{aligned}
\sigma_j - \varepsilon &= \frac{h^2 W_j}{6}\mu(\rho_j) - \varepsilon \\
&= \frac{h^2 W_j}{6}\left(\frac{1 + 2\cosh^2(\rho_j h/2)}{2\sinh^2(\rho_j h/2)}\right) - \varepsilon \\
&= \frac{h^2 W_j}{6}\left(1 + \frac{3}{2\sinh^2(\rho_j h/2)}\right) - \varepsilon \\
&= \frac{h^2 W_j}{6} + \left[\frac{h^2 W_j}{4\sinh^2(\rho_j h/2)} - \varepsilon\right] \\
&= \frac{h^2 W_j}{6} + \varepsilon\left[\frac{(h\rho_j/2)^2}{\sinh^2(\rho_j h/2)} - 1\right].
\end{aligned}$$

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This gives

$$| \sigma_j - \varepsilon | \leq Ch^2. \tag{6.2}$$

Therefore for all ε, we have from (6.1) and (6.2)

$$| \sigma_j - \varepsilon | \leq Ch^2. \ \square$$

LEMMA 6.2. *The B-splines $B_{-1}, B_0, \ldots, B_{N+1}$ satisfy the inequality*

$$\sum_{i=-1}^{N+1} | B_i(x) | \leq 10, \qquad 0 \leq x \leq 1.$$

Proof. The lemma is shown in [6]. It has previously been applied in [19]. $\square$

THEOREM 6.3. *Let $W(x), Z(x) \in C^2[0,1]$ and $W(x) \geq W^* > 0$. The collocation approximation $S(x)$ from the space $\phi_3(\Omega)$ to the solution $V(x)$ of the boundary value problem (3.6) and (3.7) exists and the error bound is given by the following relation*

$$\|V - S\| \leq Ch^2,$$

for sufficiently small h and a positive generic constant C.

Proof. To estimate the error $\|V(x) - S(x)\|$, let Y_n be the unique spline interpolant from $\Phi_3(\bar{\Omega}_N)$ to the solution $V(x)$ of our boundary value problem (3.6) and (3.7). Since $Z(x) \in C^2(\bar{\Omega})$, $V(x) \in C^4(\bar{\Omega})$ and it follows from de Boor-Hall error estimates that

$$\|D^j(V(x) - Y_n)\| \leq \lambda_j h^{4-j}, \quad j = 0, 1, 2, \tag{6.3}$$

where h is the uniform mesh spacing and C is a constant independent of h and N. Let

$$Y_n(x) = \sum_{i=-1}^{N+1} b_i B_i(x).$$

It follows immediately from the estimates (6.3) that

$$| L_\sigma S(x_i) - L_\sigma Y_n(x_i) | \leq \omega h^2, \tag{6.4}$$

where $\omega = [\Upsilon \lambda_2 + \lambda_0 \|W\| h^2]$, with $\Upsilon = \max_i \sigma_i$. Let $L_\sigma Y_n(x_i) = \hat{Z}_n(x_i) \ \forall \ i$ and $\hat{Z}^n = (\hat{Z}_n(x_0), \hat{Z}_n(x_1), \ldots, \hat{Z}_n(x_N))^t$. Now it is clear from (6.4) that the ith coordinate $[T(c^N - b^N)]_i$ of $T(c^N - b^N)$, where $b^N = (b_0, b_1, \ldots, b_N)^t$, satisfies the inequality

$$| [T(c^N - b^N)]_i | \leq \omega h^2. \tag{6.5}$$

Since $(Tc^N)_i = Z(x_i)$ and $(Tb^N)_i = \hat{Z}_n(x_i) \ \forall \ i = 0(1)N$. The ith coordinate of $[T(c^N - b^N)]$ is the ith equation,

$$(-6\sigma_i/h^2 + W_i)\delta_{i-1} + (12\sigma_i/h^2 + 4W_i)\delta_i + (-6\sigma_i/h^2 + W_i)\delta_{i+1} = \xi_i, \tag{6.6}$$

for $1 \leq i \leq N - 1$, where

$$\delta_i = b_i - c_i, \ -1 \leq i \leq N + 1 \text{ and } \quad \xi_i = Z(x_i) - \hat{Z}_n(x_i), \ 1 \leq i \leq N - 1.$$

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Obviously, $\mid \xi_i \mid \leq \omega h^2$. Let $\tilde{\xi} = \max_{1\leq i\leq N-1} \mid \xi_i \mid$. Also consider $\delta = (\delta_{-1}, \delta_0, \dots, \delta_{N+1})^t$ and define $e_i = \mid \delta_i \mid$ and $\tilde{e} = \max_{1\leq i\leq N} \mid e_i \mid$. Taking absolute value for h sufficiently small, and using the fact that $0 < W^* \leq W(x) \;\; \forall\, x \in \bar{\Omega}$, we have from (6.6),

$$(12\sigma_i/h^2 + 4W^*)\tilde{e} \leq \tilde{\xi} + 2\tilde{e}(6\sigma_i/h^2 + W^*).$$

This gives

$$\tilde{e} \leq \frac{\omega h^2}{2W^*}. \tag{6.7}$$

Now to estimate e_{-1}, e_0, e_N and e_{N+1}, we observe that the first and last equation of the the system $T(c^N - b^N) = (Z^n - \hat{Z}^n)$ where $Z^n = (Z_0, Z_1, \dots, Z_N)$, gives

$$e_0 \leq \frac{\omega h^4}{36\sigma_0} \qquad \text{and} \qquad e_N \leq \frac{\omega h^4}{36\sigma_N}.$$

Now e_{-1} and e_{N+1} can be evaluated using boundary conditions (4.5) and (4.6) as

$$e_{-1} \leq \frac{\omega h^4}{9\sigma_0} + \frac{\omega h^2}{2W^*} \qquad \text{and} \qquad e_{N+1} \leq \frac{\omega h^4}{9\sigma_N} + \frac{\omega h^2}{2W^*}.$$

Therefore, using the value $\omega = [\Upsilon\lambda_2 + \lambda_0\|W\|h^2]$, we get

$$e = \max_{-1\leq i\leq N+1}\{e_i\} \leq Ch^2, \quad \text{provided} \quad h^6/\sigma_0 \sim 0, \quad h^6/\sigma_N \sim 0. \tag{6.8}$$

Now we have

$$S(x) - Y_n(x) = \sum_{i=-1}^{N+1} (c_i - b_i)B_i(x). \tag{6.9}$$

Thus using (6.8) and Lemma 6.2, we get

$$\|S - Y_n\| \leq Ch^2. \tag{6.10}$$

Now using (6.3) and (6.10), we obtain

$$\|V - S\| \leq Ch^2$$

as required. □

7. Test examples and numerical results. To demonstrate the efficiency of the method, several examples having boundary layer at one or both end points has been analyzed. For each ε and N, the maximum absolute errors at nodal points are approximated by the formula

$$E_\varepsilon^N = \max_{0\leq j\leq N} \mid y(x_j) - y_j \mid,$$

where $y(x_j)$ and y_j are the exact and computed solution of the given problem and nodal points x_j. Also for each N the ε-uniform error at nodal point is approximated by $E^N = \max_{\varepsilon} E_\varepsilon^N$.

EXAMPLE 7.1. First we consider the problem [6],

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x), \quad y(0) = 0, \quad y(1) = 0.$$

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TABLE 7.1
Maximum absolute error for Example 7.1, without fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
$k = 4$	7.10E-03	1.78E-03	4.45E-04	1.11E-04	2.78E-05	6.96E-06	1.74E-06	4.35E-07
8	1.64E-02	3.80E-03	9.34E-04	2.32E-04	5.80E-05	1.45E-05	3.63E-06	9.07E-07
12	1.52E-01	6.35E-02	1.69E-02	3.92E-03	9.63E-04	2.40E-04	5.99E-05	1.50E-05
16	2.57E-01	2.28E-01	1.52E-01	6.35E-02	1.69E-02	3.92E-03	9.64E-04	2.40E-04
20	2.67E-01	2.65E-01	2.57E-01	2.28E-01	1.52E-01	6.35E-02	1.69E-02	3.92E-03
24	2.68E-01	2.68E-01	2.67E-01	2.65E-01	2.57E-01	2.28E-01	1.52E-01	6.35E-02
E^N	2.68E-01	2.68E-01	2.67E-01	2.65E-01	2.57E-01	2.28E-01	1.52E-01	6.35E-02

TABLE 7.2
Maximum absolute error for Example 7.1, with fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
$k = 4$	8.10E-03	2.03E-03	5.08E-04	1.27E-04	3.18E-05	7.94E-06	1.99E-06	4.96E-07
8	6.69E-03	1.62E-03	4.03E-04	1.01E-04	2.51E-05	6.28E-06	1.57E-06	3.92E-07
12	9.41E-03	1.88E-03	4.21E-04	1.02E-04	2.52E-05	6.28E-06	1.57E-06	3.92E-07
16	1.24E-02	2.91E-03	5.93E-04	1.18E-04	2.63E-05	6.35E-06	1.57E-06	3.92E-07
20	1.27E-02	3.18E-03	7.84E-04	1.82E-04	3.71E-05	7.36E-06	1.64E-06	3.97E-07
24	1.27E-02	3.20E-03	8.01E-04	2.00E-04	4.90E-05	1.14E-05	2.32E-06	4.60E-07
E^N	1.27E-02	3.20E-03	8.01E-04	2.00E-04	4.90E-05	1.14E-05	2.32E-06	4.60E-07

The exact solution is given by

$$\frac{\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})}{1+\exp(-1/\sqrt{\varepsilon})} - \cos^2(\pi x).$$

EXAMPLE 7.2. Now consider the following singular perturbation problem [16],

$$-\varepsilon y'' + \frac{4}{(x+1)^4}[1+\sqrt{\varepsilon}(x+1)]y = f(x), \quad y(0) = 2, \quad y(1) = -1,$$

where

$$f(x) = -\frac{4}{(x+1)^4} \times \left[\{1+\sqrt{\varepsilon}(x+1) + 4\pi^2\varepsilon\}\cos\left(\frac{4\pi x}{x+1}\right) - 2\pi\varepsilon(x+1)\sin\left(\frac{4\pi x}{x+1}\right) + \frac{3\{1+\sqrt{\varepsilon}(x+1)\}}{\exp(1/\sqrt{\varepsilon})-1}\right].$$

Its exact solution is given by

$$y(x) = -\cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\{\exp(-2x/\sqrt{\varepsilon}(x+1)) - \exp(-1/\sqrt{\varepsilon})\}}{1-\exp(-1/\sqrt{\varepsilon})}.$$

EXAMPLE 7.3. Finally consider the problem [16],

$$-\varepsilon[(1+x^2)y']' + \left[\frac{\cos x}{(3-x)^3}\right] y = 4(3x^2-3x+1)[(x-1/2)^2+2], \quad y(0) = -1, \quad y(1) = 0.$$

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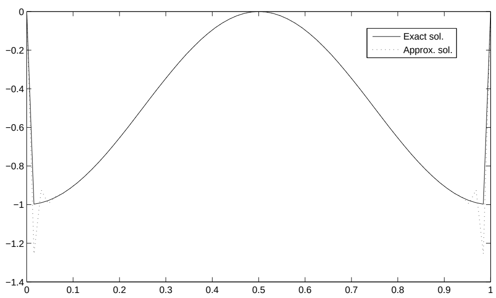

FIG. 7.1. *Exact and approximate solutions of Example 7.1, without fitting factor.*

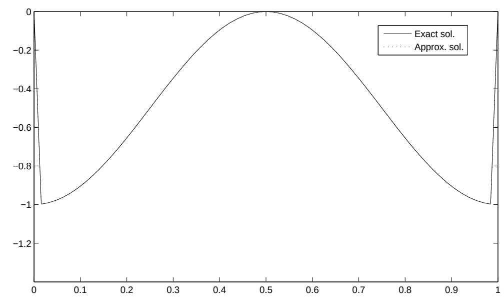

FIG. 7.2. *Exact and approximate solutions of Example 7.1 with fitting factor.*

TABLE 7.3
Maximum absolute error for Example 7.2, without fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
$k = 4$	4.56E-02	1.13E-02	2.85E-03	7.12E-04	1.78E-04	4.45E-05	1.11E-05	2.78E-06
8	2.47E-01	6.31E-02	1.44E-02	3.53E-03	8.78E-04	2.19E-04	5.48E-05	1.37E-05
12	8.57E-01	5.10E-01	2.04E-01	5.37E-02	1.24E-02	3.05E-03	7.59E-04	1.90E-04
16	1.01E+00	8.71E-01	7.25E-01	4.70E-01	1.94E-01	5.14E-02	1.19E-02	2.93E-03
20	1.02E+00	9.07E-01	8.46E-01	7.96E-01	6.94E-01	4.60E-01	1.91E-01	5.08E-02
24	1.02E+00	9.09E-01	8.55E-01	8.27E-01	8.08E-01	7.78E-01	6.86E-01	4.58E-01
E^N	1.02E+00	9.09E-01	8.55E-01	8.27E-01	8.08E-01	7.78E-01	6.86E-01	4.58E-01

The exact solution for this problem is not available. Since the exact solution does not exist, the rate of convergence and maximum absolute error based on the double-mesh principle [20] has been given in Tables 7.7 and 7.8.

$$E_\varepsilon^N = \max_{0 \le j \le N} \mid y_j^N - y_{2j}^{2N} \mid, \qquad r_k = \log_2 \left(\frac{z_k}{z_{k+1}} \right), \quad k = 0, 1, 2, \dots .$$

where

$$z_k = \max_j \mid y_j^{h/2^k} - y_{2j}^{h/2^{k+1}} \mid, \quad k = 0, 1, 2 \dots .$$

and $y_j^{h/2^k}$ denotes the value of y_j for the mesh length $h/2^k$.

8. Discussions and conclusions. We have described a B-spline collocation method for the solution of self-adjoint singularly perturbed two-point boundary value problems. It is a practical method and easily can be implemented on a computer. The method has been analyzed for parameter-uniform convergence. For Example 7.3, we have computed the rate of convergence (see Table 7.8) which shows uniform second-order convergence as predicted by the theory.

Graphs have been plotted for Examples 7.1 and 7.2 for values of $x \in [0, 1]$ versus the computed solution obtained at different values of x for a fixed ε. For each plot, we took $N = 64$ and $\varepsilon = 2^{-20}$. For Example 7.3, the exact solution is not available. We show two graphs, with and without fitting factor. For each graph, we use $\varepsilon = 2^{-20}$ and $N = 32,\ N = 64$.

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TABLE 7.4
Maximum absolute error for Example 7.2, with fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
$k = 4$	3.96E-02	1.01E-02	2.53E-03	6.34E-04	1.59E-04	3.96E-05	9.91E-06	2.48E-06
8	2.18E-02	6.93E-03	1.67E-03	4.16E-04	1.04E-04	2.60E-05	6.49E-06	1.62E-06
12	4.47E-02	8.62E-03	6.34E-03	2.54E-03	6.13E-04	1.51E-04	3.79E-05	9.46E-06
16	5.04E-02	1.91E-02	5.03E-03	9.73E-04	2.09E-03	7.23E-04	1.75E-04	4.31E-05
20	5.07E-02	1.96E-02	5.69E-03	1.49E-03	3.45E-04	4.06E-04	5.60E-04	1.87E-04
24	5.07E-02	1.96E-02	5.73E-03	1.52E-03	3.89E-04	9.68E-05	2.25E-05	1.15E-04
E^N	5.07E-02	1.96E-02	6.34E-03	2.54E-03	2.09E-03	7.23E-04	5.60E-04	1.87E-04

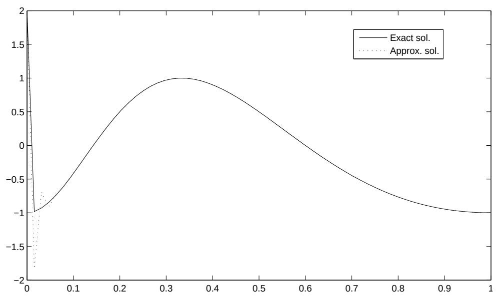

FIG. 7.3. *Exact and approximate solutions of Example 7.2, without fitting factor.*

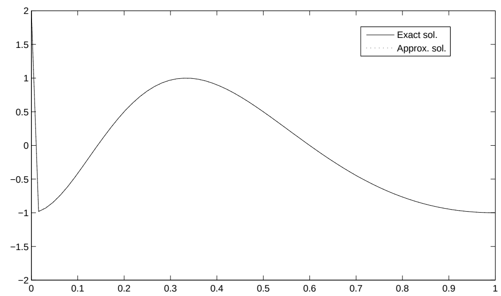

FIG. 7.4. *Exact and approximate solutions of Example 7.2, with fitting factor.*

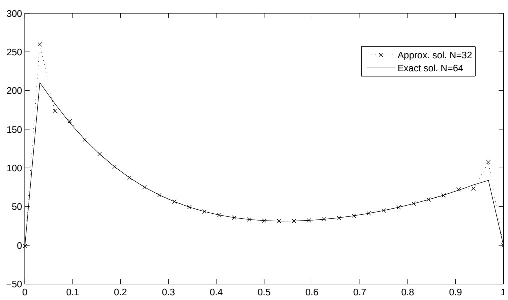

FIG. 7.5. *Exact and approximate solutions of Example 7.3, without fitting factor.*

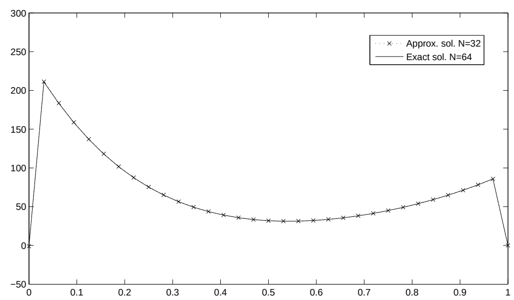

FIG. 7.6. *Exact and approximate solutions of Example 7.3, with fitting factor.*

For Examples 7.1 and 7.2, it can be seen in Figures 7.1 and 7.3 that the exact and approximate solutions without using fitting factors deviate from each other in the boundary layer regions for smaller ε. To control these fluctuations, we used the fitting-factor technique and the resulting behavior of these solutions can be seen in Figures 7.2 and 7.4. Also, for Example 7.3, it can be easily seen that the deviation in the graph (Figure 7.6) of the approximate solution using a fitting factor corresponding to two values of N $(32$ and $64)$ is much smaller than the corresponding deviation in the graph (Figure 7.5) of the approximate solution without a fitting factor corresponding to these values of N.

We have replaced ε by σ_j in the normalized form and not in the original self-adjoint

TABLE 7.5
Comparisons of maximum absolute errors for Example 7.2 with those in [16].

N	Results in [16]			Our results		
	$\varepsilon = 1$	$\varepsilon = (1/N)^{0.5}$	$\varepsilon = (1/N)^{1.0}$	$\varepsilon = 1$	$\varepsilon = (1/N)^{0.5}$	$\varepsilon = (1/N)^{1.0}$
8	4.2E-01	3.8E-01	3.3E-01	2.0E-01	1.9E-01	1.7E-01
16	1.1E-01	9.5E-02	7.8E-02	5.4E-02	4.8E-02	4.0E-02
32	2.7E-02	2.3E-02	1.8E-02	1.4E-02	1.2E-02	8.9E-03
64	6.9E-03	5.6E-03	4.2E-03	3.5E-03	2.8E-03	1.9E-03
128	1.7E-03	1.3E-03	1.0E-03	8.6E-04	6.7E-04	3.8E-04
256	4.3E-04	3.1E-04	2.5E-04	2.2E-04	1.6E-04	1.0E-04
512	1.1E-04	7.4E-05	6.3E-05	5.4E-05	3.7E-05	4.1E-05

TABLE 7.6
Maximum absolute error for Example 7.3, without fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N					
	32	64	128	256	512	1024
$k = 2$	1.39E-03	3.47E-04	8.67E-05	2.17E-05	5.42E-06	1.35E-06
4	5.34E-03	1.34E-03	3.34E-04	8.35E-05	2.09E-05	5.22E-06
6	1.86E-02	4.65E-03	1.16E-03	2.91E-04	7.27E-05	1.82E-05
8	4.79E-02	1.20E-02	2.99E-03	7.48E-04	1.87E-04	4.68E-05
10	1.20E-01	2.99E-02	7.46E-03	1.86E-03	4.66E-04	1.17E-04
12	4.17E-01	1.04E-01	2.60E-02	6.49E-03	1.62E-03	4.06E-04
14	1.68E+00	4.08E-01	1.02E-01	2.54E-02	6.34E-03	1.59E-03
16	7.57E+00	1.70E+00	4.10E-01	1.02E-01	2.55E-02	6.38E-03
18	2.39E+01	7.65E+00	1.71E+00	4.13E-01	1.03E-01	2.57E-02
20	4.94E+01	2.42E+01	7.70E+00	1.72E+00	4.15E-01	1.04E-01
22	6.91E+01	5.01E+01	2.43E+01	7.73E+00	1.73E+00	4.17E-01
24	7.72E+01	7.01E+01	5.04E+01	2.44E+01	7.75E+00	1.73E+00
26	7.95E+01	7.83E+01	7.06E+01	5.06E+01	2.45E+01	7.76E+00
28	8.02E+01	8.08E+01	7.89E+01	7.08E+01	5.07E+01	2.45E+01
E^N	8.02E+01	8.08E+01	7.89E+01	7.08E+01	5.07E+01	2.45E+01

problem, with $a(x) \neq$ constant, because in that case ε is a multiple of both the second and first derivative terms, which will cause ill-conditioning in the tridiagonal system. In normalized form, ε multiplies the second derivative term only. Hence, the fitting-factor technique for the normalized form can be implemented easily. This shows the importance of reducing the original self-adjoint problem to normal form.

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TABLE 7.7
Maximum absolute error for Example 7.3, with fitting factor.

$\varepsilon = 2^{-k}$	Number of mesh points N					
	32	64	128	256	512	1024
$k = 2$	1.31E-03	3.28E-04	8.21E-05	2.05E-05	5.13E-06	1.28E-06
4	4.93E-03	1.23E-03	3.08E-04	7.71E-05	1.93E-05	4.82E-06
6	1.60E-02	4.00E-03	1.00E-03	2.50E-04	6.26E-05	1.56E-05
8	3.71E-02	9.27E-03	2.32E-03	5.79E-04	1.45E-04	3.62E-05
10	6.19E-02	1.54E-02	3.86E-03	9.65E-04	2.41E-04	6.03E-05
12	9.39E-02	2.34E-02	5.83E-03	1.46E-03	3.64E-04	9.10E-05
14	1.34E-01	3.29E-02	8.15E-03	2.03E-03	5.08E-04	1.27E-04
16	1.90E-01	4.31E-02	1.05E-02	2.60E-03	6.50E-04	1.62E-04
18	2.84E-01	7.89E-02	2.56E-02	6.20E-03	1.54E-03	3.84E-04
20	4.08E-01	7.82E-02	5.26E-02	1.49E-02	3.61E-03	9.05E-04
22	4.60E-01	1.06E-01	2.78E-02	2.97E-02	7.97E-03	1.94E-03
24	4.62E-01	1.21E-01	2.78E-02	1.91E-02	1.57E-02	4.12E-03
26	4.62E-01	1.21E-01	3.08E-02	7.09E-03	1.09E-02	8.09E-03
28	4.62E-01	1.21E-01	3.10E-02	7.80E-03	1.79E-03	5.78E-03
E^N	4.62E-01	1.21E-01	5.26E-02	2.97E-02	1.57E-02	8.09E-03

TABLE 7.8
Rate of convergence for Example 7.3.

$\varepsilon = 2^{-k}$	r(0)	r(1)	r(2)	r(3)	r(4)
$k = 2$	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
4	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
6	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
8	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
10	2.01E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
12	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
14	2.03E+00	2.01E+00	2.01E+00	2.00E+00	2.00E+00
16	2.14E+00	2.04E+00	2.01E+00	2.00E+00	2.00E+00

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