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MINIMAL GERSCHGORIN SETS FOR PARTITIONED MATRICES III. SHARPNESS OF BOUNDARIES AND MONOTONICITY AS A FUNCTION OF THE PARTITION*

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Dedicated to Olga Taussky and John Todd, on the occasion of their important birthdays in 1996, for their inspiring work in matrix theory and numerical analysis.

Abstract. Making use, from the preceding paper, of the affirmative solution of the Spectral Conjecture, it is shown here that the general boundaries, of the minimal Gerschgorin sets for partitioned matrices, are sharp, and that monotonicity of these minimal Gerschgorin sets, as a function of the partitionings, is obtained. These results extend and sharpen an earlier paper from 1970 on this topic.

Key words. minimal Gerschgorin sets, partitioned matrices, monotonicity.

AMS subject classification. 15A18.

1. Background and Notations. For brevity, we use the notations of the previous paper [4], so that for n a positive integer, a *partition* π of $\mathbb{C}^n$ is a representation of $\mathbb{C}^n$ as a direct sum of N pairwise disjoint nonempty linear subspaces (cf. [4, eq. (1.1)]):

$$\mathbb{C}^n = W_1 \dot{+} W_2 \dot{+} \cdots \dot{+} W_N. \tag{1.1}$$

Here, π is denoted by $\pi := \{r_j\}_{j=0}^N$, with the nonnegative integers $\{r_j\}_{j=0}^N$ satisfying $r_0 := 0 < r_1 < \cdots < r_N := n$, where it is assumed, without essential loss of generality, that

$$W_j = span\,\{\mathbf{e}_k : r_{j-1}+1 \le k \le r_j\} \quad (j = 1, 2, \cdots, N), \tag{1.2}$$

the vectors $\{\mathbf{e}_j\}_{j=1}^n$ being the standard column basis vectors for $\mathbb{C}^n$. Then, given a matrix A in $\mathbb{C}^{n,n}$ and given a partition $\pi = \{r_j\}_{j=0}^N$ of $\mathbb{C}^n$, the matrix A is partitioned, with respect to π, as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & & & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{bmatrix} = [A_{i,j}] \qquad (i, j = 1, 2, \cdots, N), \tag{1.3}$$

where each submatrix $A_{i,j}$ represents a linear transformation from W_j into W_i. With (1.3), we define

$$D_\pi := diag\,\{A_{1,1};\ A_{2,2};\ \cdots;\ A_{N,N}\} \tag{1.4}$$

as the *block-diagonal* matrix of A, with respect to the partition π. As in [4, Section 1], associated with each norm N-tuple

$$\phi := (\phi_1, \phi_2, \cdots, \phi_N)$$

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(where each ϕ_j is a vector norm on W_j) is a vector norm on $\mathbb{C}^n$, defined by

$$\|\mathbf{x}\|_\phi := \max_{1\le j\le N} \{\phi_j(P_j\mathbf{x})\} \qquad (\mathbf{x}\in\mathbb{C}^n)$$

(where each P_j is the projection operator from $\mathbb{C}^n$ to W_j), and, for any matrix $B \in \mathbb{C}^{n,n}$, its induced operator norm, associated with ϕ, is defined by

$$\|B\|_\phi := \sup_{\|\mathbf{x}\|_\phi = 1} \|B\mathbf{x}\|_\phi.$$

The collection of all such norm N-tuples $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$ is then denoted by Φ_π.

Fix a matrix A in $\mathbb{C}^{n,n}$ and fix a partition π of $\mathbb{C}^n$. Using a norm argument introduced by Householder (cf. [1, p. 66]) to obtain inclusion sets for the eigenvalues of a matrix, let $\sigma(A)$ denote the spectrum of A, and consider any $\lambda \in \sigma(A)$. Then, there is an $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{C}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$, and this can also be expressed as

$$(A - D_\pi)\mathbf{x} = (\lambda I - D_\pi)\mathbf{x},$$

where I denotes the identity matrix in $\mathbb{C}^{n,n}$. If $\lambda \notin \sigma(D_\pi)$, this implies that

$$(\lambda I - D_\pi)^{-1}(A - D_\pi)\mathbf{x} = \mathbf{x}, \tag{1.5}$$

which, from the above norm definitions, gives

$$\|(\lambda I - D_\pi)^{-1}(A - D_\pi)\|_\phi \geq 1 \quad (\text{ any } \phi \in \Phi_\pi). \tag{1.6}$$

Now for $\phi \in \Phi_\pi$, set

$$H_\pi^\phi(A) := \left\{z \in \mathbb{C} : z \notin \sigma(D_\pi) \text{ and } \|(zI - D_\pi)^{-1}(A - D_\pi)\|_\phi \geq 1\right\}, \tag{1.7}$$

$$G_\pi^\phi(A) := H_\pi^\phi(A) \cup \sigma(D_\pi), \tag{1.8}$$

and

$$G_\pi(A) := \bigcap_{\phi\in\Phi_\pi} G_\pi^\phi(A). \tag{1.9}$$

The set $G_\pi(A)$ is called the *minimal Gerschgorin set* for the matrix A in $\mathbb{C}^{n,n}$, relative to the partition π. From (1.7)-(1.9), it is evident that $G_\pi(A)$ can also be expressed as

$$G_\pi(A) = H_\pi(A) \cup \sigma(D_\pi), \tag{1.10}$$

where

$$\begin{aligned} H_\pi(A) &:= \bigcap_{\phi\in\Phi_\pi} H_\pi^\phi(A) \\ &= \left\{ z \in \mathbb{C} : z \notin \sigma(D_\pi) \text{ and } \inf_{\phi\in\Phi_\pi} \|(zI - D_\pi)^{-1}(A - D_\pi)\|_\phi \geq 1 \right\}. \end{aligned} \tag{1.11}$$

We remark that the minimal Gerschgorin set $G_\pi(A)$, as given above in (1.9) or (1.10), differs slightly from the definition in [7] of a minimal Gerschgorin set, because the definition used in [7] uses a somewhat complicated subset of $\sigma(D_\pi)$, rather than $\sigma(D_\pi)$ itself, as in (1.10) above.

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We next claim that any z in $H_\pi^\phi(A)$ satisfies

$$|z| \le \|A - D_\pi\|_\phi + \|D_\pi\|_\phi \quad (\text{any } \phi \in \Phi_\pi), \tag{1.12}$$

for if $|z| \le \|D_\pi\|_\phi$, the above inequality is trivially satisfied. If $|z| > \|D_\pi\|_\phi$, we have, with (1.7), that

$$\begin{aligned}\|A - D_\pi\|_\phi &= \|(zI - D_\pi)\cdot\left\{(zI-D_\pi)^{-1}(A-D_\pi)\right\}\|_\phi \\ &= \|z\left\{(zI-D_\pi)^{-1}(A-D_\pi)\right\} - D_\pi\left\{(zI-D_\pi)^{-1}(A-D_\pi)\right\}\|_\phi \\ &\ge (|z| - \|D_\pi\|)\cdot\|(zI-D_\pi)^{-1}(A-D_\pi)\|_\phi \ge |z| - \|D_\pi\|_\phi,\end{aligned}$$

which gives (1.12). Thus, (1.12) shows that G_π^ϕ is a *bounded* set in $\mathbb{C}$. Also, it is evident from (1.7) and (1.8) that $G_\pi^\phi(A)$ is *closed*. Thus, from (1.11), $G_\pi(A)$ is also closed and bounded in $\mathbb{C}$.

For the fixed matrix A in $\mathbb{C}^{n,n}$ partitioned by π as in (1.3), we define its associated *equimodular class* $\Omega_\pi(A)$ (cf. [7, Definition 3]) as

$$\begin{aligned}\Omega_\pi(A) := \{\, B = [B_{k,\ell}] \in \mathbb{C}^{n,n} : B_{k,k} = A_{k,k}, &\text{ and for each pair } (k,\ell) \text{ with} \\ k \ne \ell, B_{k,\ell} = e^{i\theta_{k,\ell}}A_{k,\ell} &\text{ for some real } \theta_{k,\ell} \quad (k,\ell = 1,2,\cdots,N)\}.\end{aligned} \tag{1.13}$$

With $\sigma(\Omega_\pi(A)) := \bigcup_{B\in\Omega_\pi(A)} \sigma(B)$, we next establish

THEOREM 1.1. *Let π be a partition of $\mathbb{C}^n$ and let A in $\mathbb{C}^{n,n}$ be partitioned by π. Then,*

$$\sigma(\Omega_\pi(A)) \subset G_\pi(A). \tag{1.14}$$

Proof. We first show that $\sigma(A) \subset G_\pi(A)$. Consider any $\lambda \in \sigma(A)$. If $\lambda \in \sigma(D_\pi)$, then $\lambda \in G_\pi(A)$ from (1.10). If $\lambda \notin \sigma(D_\pi)$, let $\mathbf{x} \ne \mathbf{0}$ be such that $A\mathbf{x} = \lambda\mathbf{x}$ which gives

$$(A - D_\pi)\mathbf{x} = (\lambda I - D_\pi)\mathbf{x}.$$

As $\lambda \notin \sigma(D_\pi)$, it follows, as in (1.5) and (1.6), that

$$\|(\lambda I - D_\pi)^{-1}(A - D_\pi)\|_\phi \ge 1 \quad (\text{ any } \phi \in \Phi_\pi);$$

whence,

$$\inf_{\phi\in\Phi_\pi} \|(\lambda I - D_\pi)^{-1}(A - D_\pi)\|_\phi \ge 1.$$

From (1.11), we deduce that $\lambda \in H_\pi(A)$, and, consequently from (1.10), also that $\lambda \in G_\pi(A)$. Thus,

$$\sigma(A) \subset G_\pi(A). \tag{1.15}$$

Next, for any $z \notin \sigma(D_\pi)$, consider the matrix $(zI - D_\pi)^{-1}(A - D_\pi)$, which, when partitioned by π, is written as

$$(zI - D_\pi)^{-1}(A - D_\pi) =: [E_{i,j}(z)] \quad (i, j = 1, 2, \cdots N).$$

Because D_π is the block-diagonal matrix of (1.4), it directly follows that the block-diagonal submatrices $E_{i,i}(z)$ of $(zI - D_\pi)^{-1}(A - D_\pi)$ satisfy

$$E_{i,i}(z) = O \quad (i = 1, 2, \cdots, N),$$

i.e., in the notation of the previous paper (cf. [4, eq. (1.7)]), $(zI - D_\pi)^{-1}(A - D_\pi)$ is a π-*invertebrate matrix*. From this and from the definition in (1.13), it can be verified that an application of [7, Theorem 2] gives us that

$$\|(zI - D_\pi)^{-1}(A - D_\pi)\|_\phi = \|(zI - D_\pi)^{-1}(B - D_\pi)\|_\phi \quad (\text{all } \phi \in \Phi_\pi,\ \text{all } B \in \Omega_\pi(A))\,.$$

But then, we directly see from (1.11) that

$$H_\pi(A) = H_\pi(B) \quad (\text{any } B \in \Omega_\pi(A))\,,$$

and, by definition (1.13), that the block-diagonal matrix of any B in $\Omega_\pi(A)$ is D_π. Hence (cf. (1.10))

$$G_\pi(A) = G_\pi(B) \quad (\text{any } B \in \Omega_\pi(A))\,. \tag{1.16}$$

Applying (1.15) to any B in $\Omega_\pi(A)$ gives, with (1.16),

$$\sigma(B) \subset G_\pi(B) = G_\pi(A),$$

which yields the desired result that $\sigma\left(\Omega_\pi(A)\right) \subset G_\pi(A)$. $\square$

The inclusion (1.14) of Theorem 1.1 is interesting, in that it raises the question of just how sharp this inclusion is. As we shall see in Section 2, this inclusion is indeed sharp in a precise sense.

2. The Sharpness of the Boundary of the Minimal Gerschgorin Set. The object of this section is to establish Theorem 2.1, below. For notation, if T is any set in the complex plane $\mathbb{C}$, then T' denotes its complement and $\bar{T}$ its closure (in the usual topology of $\mathbb{C}$). The boundary of $G_\pi(A)$ is then defined by

$$\partial G_\pi(A) := \overline{G_\pi(A)} \bigcap \overline{G'_\pi(A)} = G_\pi(A) \bigcap \overline{G'_\pi(A)}, \tag{2.1}$$

the last equality holding since $G_\pi(A)$ is a closed set from our discussion in Section 1. In addition, *int* T denotes the interior of a set T i.e., *int* $T := T \backslash \partial T$. With this notation, we state our next result as

THEOREM 2.1. *Let π be a partition of $\mathbb{C}^n$, and let A in $\mathbb{C}^{n,n}$ be partitioned by π. Then, the inclusion of (1.14) of Theorem 1.1 for the minimal Gerschgorin set $G_\pi(A)$ is sharp in the sense that*

$$\partial G_\pi(A) \subset \sigma\left(\Omega_\pi(A)\right) \subset G_\pi(A). \tag{2.2}$$

In other words, each point of the boundary $\partial G_\pi(A)$ is an eigenvalue of some matrix B in the equimodular class $\Omega_\pi(A)$ of A.

To prove Theorem 2.1, we consider the two functions

$$\eta(z) := \inf_{\phi \in \Phi_\pi} \|(zI - D_\pi)^{-1}(A - D_\pi)\|_\phi \quad (z \notin \sigma(D_\pi))\,, \tag{2.3}$$

and

$$\nu(z) := \sup_{B \in \Omega_\pi(A)} \rho\left\{(zI - D_\pi)^{-1}(B - D_\pi)\right\} \quad (z \notin \sigma(D_\pi)), \tag{2.4}$$

where $\rho(F)$ denotes in general the spectral radius of a square matrix F (i.e., $\rho(F) := \max\{|\lambda| : \ \lambda \in \sigma(F)\}$). Noting again that the matrix $(zI - D_\pi)^{-1}(A - D_\pi)$ is π-invertebrate for any $z \notin \sigma(D_\pi)$, we immediately deduce, from Theorem 2.1 of the previous paper [4], that

$$\eta(z) = \nu(z) \quad (z \notin \sigma(D_\pi)). \tag{2.5}$$

And, as shown in [7, Lemma 2], $\nu(z)$, and hence $\eta(z)$, is continuous in $\mathbb{C}\backslash\sigma(D_\pi)$. We also note, from (1.11), that we can now express the set $H_\pi(A)$ as

$$H_\pi(A) = \{z \in \mathbb{C} : z \notin \sigma(D_\pi) \text{ and } \eta(z) \geq 1\}. \tag{2.6}$$

We next give a useful result which is a less precise version of a result of Householder, Varga, and Wilkinson [2, Theorem 2]. (For completeness, we include its short proof.)

LEMMA 2.2. *Let $\lambda \in \sigma(D_\pi)$. If the matrix $(zI - D_\pi)^{-1}(A - D_\pi)$ remains bounded (in every entry) as $z \to \lambda$, then $\lambda \in \sigma(A)$.*

Proof. As in [2], let $\mathbf{x}^H \neq \mathbf{0}$ be a left eigenvector of D_π, corresponding to λ. Then, $\lambda \mathbf{x}^H = \mathbf{x}^H D_\pi$, which we can write as $(z - \lambda)\mathbf{x}^H = \mathbf{x}^H(zI - D_\pi)$. If $z \notin \sigma(D_\pi)$, this gives

$$\mathbf{x}^H(zI - D_\pi)^{-1} = (z - \lambda)^{-1}\mathbf{x}^H,$$

and, on post-multiplying the above by the matrix $A - D_\pi$, we have

$$\mathbf{x}^H(zI - D_\pi)^{-1}(A - D_\pi) = (z - \lambda)^{-1}\mathbf{x}^H(A - D_\pi). \tag{2.7}$$

If the matrix $(zI - D_\pi)^{-1}(A - D_\pi)$ remains bounded as $z \to \lambda$, it is evident from (2.7) that $\mathbf{x}^H(A - D_\pi) = \mathbf{0}$. Hence,

$$\mathbf{x}^H A = \mathbf{x}^H D_\pi = \lambda \mathbf{x}^H,$$

and $\lambda \in \sigma(A)$. □

With Lemma 2.2, we next establish

LEMMA 2.3. *Let $\lambda \in \sigma(D_\pi)$. If $\lambda \notin$ int $G_\pi(A)$, then $\lambda \in \sigma(A)$.*

Proof. Consider any $\lambda \in \sigma(D_\pi)$ with $\lambda \notin$ *int* $G_\pi(A)$. From Lemma 2.2, it suffices to show that the matrix $(zI - D_\pi)^{-1}(A - D_\pi)$ remains bounded (in every element) as $z \to \lambda$. Suppose, on the contrary, that $(zI - D_\pi)^{-1}(A - D_\pi)$ is *not* bounded in every element as $z \to \lambda$. This implies that

$$\lim_{z \to \lambda} \|(zI - D_\pi)^{-1}(A - D_\pi)\|_\phi = +\infty \quad (\text{any } \phi \in \Phi_\pi).$$

Hence, from the definition in (2.3),

$$\lim_{z \to \lambda} \eta(z) = +\infty.$$

Thus, there exists a sufficiently small $\epsilon_0 > 0$ such that for every z in the punctured disk

$$\hat{\Delta}(\lambda, \epsilon_0) := \{z \in \mathbb{C} : 0 < |z - \lambda| < \epsilon_0\},$$

we have $\eta(z) \geq 1$. As a consequence of (2.6),

$$\hat{\Delta}(\lambda, \epsilon_0) \subset G_{\pi,1}(A),$$

and from (1.10), because $\lambda \in \sigma(D_\pi)$,

$$\Delta(\lambda, \epsilon_0) := \hat{\Delta}(\lambda, \epsilon_0) \cup \{\lambda\} \subset G_\pi(A).$$

But as the above inclusion gives that the entire (unpunctured) disk $\Delta(\lambda, \epsilon_0)$ lies in $G_\pi(A)$, then λ is necessarily an interior point of $G_\pi(A)$, a contradiction. Applying Lemma 2.2 then gives that $\lambda \in \sigma(A)$. $\square$

We now come to the

Proof of Theorem 2.1 Regarding the sought inclusions of (2.2), the second inclusion in (2.2) already follows from (1.14) of Theorem 1.1. Consider any $z \in \partial G_\pi(A)$, so that $z \notin int\ G_\pi(A)$. If $z \in \sigma(D_\pi)$, then from Lemma 2.3, $z \in \sigma(A) \subset \sigma(\Omega_\pi(A))$, the last inclusion following trivially from the fact (cf. (1.13)) that $A \in \Omega_\pi(A)$. Thus, if $z \in \partial G_\pi(A)$ with $z \in \sigma(D_\pi)$, then $z \in \sigma(\Omega_\pi(A))$.

If $z \in \partial G_\pi(A)$ with $z \notin \sigma(D_\pi)$, then $\eta(z)$ is well defined and from (2.6), $\eta(z) \geq 1$. On the other hand, as $z \in \partial G_\pi(A)$, we see from (2.1) that z is in the closure of $G'_\pi(A)$. This implies from (2.6) that there exists a sequence of complex numbers $\{z_k\}_{k=1}^{\infty}$ with $z_k \to z$ and with $\eta(z_k) < 1$. From the continuity of η in $\mathbb{C} \backslash \sigma(D_\pi)$, it follows that $\eta(z) = 1$. From (2.5), $\nu(z) = 1$ also, so that, from the definition of $\nu(z)$ in (2.4),

$$\nu(z) = \sup_{B \in \Omega_\pi(A)} \rho\left\{(zI - D_\pi)^{-1}(B - D_\pi)\right\} = 1. \tag{2.8}$$

From (1.13), the spectral radius of the matrix $(zI - D_\pi)^{-1}(B - D_\pi)$, for any B in $\Omega_\pi(A)$, depends continuously on $N^2 - N$ real numbers $\theta_{k,\ell}$ with $0 \leq \theta_{k,\ell} \leq 2\pi$ $(1 \leq k, \ell \leq N$ with $k \neq \ell)$. From compactness considerations, there exists a $\tilde{B} \in \Omega_\pi(A)$ for which the supremum in (2.8) is attained. Thus,

$$\rho\left\{(zI - D_\pi)^{-1}(\tilde{B} - D_\pi)\right\} = 1.$$

In fact, for a suitable multiplicative factor $e^{i\alpha}$ with α real, the matrix $\hat{B} := e^{i\alpha}(\tilde{B} - D_\pi) + D_\pi$, which is an element of $\Omega_\pi(A)$, is such that $(zI - D_\pi)^{-1}(\hat{B} - D_\pi)$ has an eigenvalue unity. Hence, there exists an $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{C}^n$ such that

$$(zI - D_\pi)^{-1}(\hat{B} - D_\pi)\mathbf{x} = \mathbf{x},$$

which implies that

$$(\hat{B} - D_\pi)\mathbf{x} = (zI - D_\pi)\mathbf{x},$$

so that

$$\hat{B}\mathbf{x} = z\mathbf{x}.$$

But as $\hat{B} \in \Omega_\pi(A)$, the above display gives that $z \in \sigma(\hat{B}) \subset \sigma(\Omega_\pi(A))$. Thus, if $z \in \partial G_\pi(A)$ with $z \notin \sigma(D_\pi)$, then $z \in \sigma(\Omega_\pi(A))$. □

Our next result considers the second inclusion of (2.2) of Theorem 2.1. By suitably extending the class $\Omega_\pi(A)$, we show that the second inclusion of (2.2) can be made one of equality. To this end, we set

$$\text{(2.9)} \qquad \begin{aligned} \hat{\Omega}_\pi(A) := \; &\{B = [B_{k,\ell}] \in \mathbb{C}^{n,n} : B_{k,k} = A_{k,k}, \text{ and there is a } \tau \in [0,1] \\ &\text{such that for each pair } (k,\ell) \text{ with } k \neq \ell, B_{k,\ell} = \tau e^{i\theta k,\ell} A_{k,\ell} \\ &\text{for some real } \; \theta_{k,\ell} \; (k,\ell = 1,2,\cdots N)\}. \end{aligned}$$

We note that the choice $\tau = 1$ in (2.9) shows that $\Omega_\pi(A) \subset \hat{\Omega}_\pi(A)$.

Lemma 2.4. *Let π be a partition of $\mathbb{C}^n$ and let A in $\mathbb{C}^{n,n}$ be partitioned by π. Then,*

$$\text{(2.10)} \qquad \sigma\left(\hat{\Omega}_\pi(A)\right) \subset G_\pi(A).$$

Proof. For any $z \in \sigma\left(\hat{\Omega}(A)\right)$, there exists a $\hat{B} \in \hat{\Omega}_\pi(A)$ and an $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{C}^n$ with $\hat{B}\mathbf{x} = z\mathbf{x}$, and this gives that

$$\text{(2.11)} \qquad \left(\hat{B} - D_\pi\right)\mathbf{x} = (zI - D_\pi)\,\mathbf{x}.$$

If $z \in \sigma(D_\pi)$, then from (1.10), $z \in G_\pi(A)$. Otherwise, if $z \notin \sigma(D_\pi)$, it follows from (2.11) that

$$\|(zI - D_\pi)^{-1}(\hat{B} - D_\pi)\|_\phi \geq 1 \quad (\text{any } \phi \in \Phi_\pi).$$

From (2.9), let $\tau \in [0,1]$ be the associated scalar factor associated with $\hat{B}$. Because $z \notin \sigma(D_\pi)$, it follows that $0 < \tau$ (for if τ were zero, $\hat{B}$ would, from (2.9), reduce exactly to D_π, giving $\sigma(\hat{B}) = \sigma(D_\pi)$, which contradicts the assumption that $z \notin \sigma(D_\pi)$). Consider next the matrix

$$B := D_\pi + \frac{1}{\tau}(\hat{B} - D_\pi).$$

It can be directly seen from (2.9) and (1.13) that $B \in \Omega_\pi(A)$, and, moreover, that

$$\|(zI - D_\pi)^{-1}(B - D_\pi)\|_\phi \geq \frac{1}{\tau} \geq 1 \quad (\text{ any } \phi \in \Phi_\pi).$$

Hence, on replacing the matrix A with B in both (2.3) and (2.6), the above inequalities give that $z \in G_\pi(B)$. But as $G_\pi(B) = G_\pi(A)$ from (1.16), the desired inclusion of (2.10) follows. □

Next, we have

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THEOREM 2.5. *Let π be a partition of $\mathbb{C}^n$ and let $A \in \mathbb{C}^{n,n}$ be partitioned by π. Then,*

$$\sigma\left(\hat{\Omega}_\pi(A)\right) = G_\pi(A). \tag{2.12}$$

Proof. Consider any $z \in G_\pi(A)$. If $z \in \sigma(D_\pi)$, then as D_π is an element of $\hat{\Omega}_\pi(A)$, as choosing $\tau = 0$ in (2.9) shows, we have $z \in \sigma\left(\hat{\Omega}(A)\right)$. Next, suppose that $z \notin \sigma(D_\pi)$, so that, from (1.10), $z \in H_\pi(A)$. From (2.5) and (2.6), we have that $t := \eta(z) = \nu(z) \geq 1$. By compactness considerations again, there exists a $B \in \Omega_\pi(A)$ and an $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{C}^n$ such that

$$(zI - D_\pi)^{-1}(B - D_\pi)\mathbf{x} = t\mathbf{x},$$

or equivalently,

$$\left\{\frac{1}{t}B + \left(1 - \frac{1}{t}\right)D_\pi\right\}\mathbf{x} = z\mathbf{x}. \tag{2.13}$$

Since, for any $\tau \in (0, 1]$ and for any $B \in \Omega_\pi(A)$, the matrix $\hat{B} := \tau B + (1 - \tau)D_\pi$ can be seen from (2.9) to be an element of $\hat{\Omega}_\pi(A)$, we deduce from (2.13) that $\hat{B}$, with $\tau := \frac{1}{t}$, is an element of $\hat{\Omega}_\pi(A)$. As z is also an eigenvalue of $\hat{B}$ from (2.13), then $z \in \sigma\left(\hat{\Omega}_\pi(A)\right)$. □

Note that we can now write the result of (2.2) of Theorem 2.1 in the form

$$\partial G_\pi(A) \subset \sigma\left(\Omega_\pi(A)\right) \subset \sigma\left(\hat{\Omega}_\pi(A)\right) = G_\pi(A). \tag{2.14}$$

3. Monotonicity. The minimal Gerschgorin set $G_\pi(A)$ of a fixed matrix A in $\mathbb{C}^{n,n}$ depends, of course, on the partition π of $\mathbb{C}^n$. The object of this section is to study this dependence on π. To this end, we recall from Section 1 that if π is a partition of $\mathbb{C}^n$, i.e., (cf. (1.1)) if

$$\mathbb{C}^n = W_1 \dot{+} W_2 \dot{+} \cdots \dot{+} W_N,$$

then the partition π can be defined in terms of the nonnegative integers $\{r_j\}_{j=0}^N$, where $r_0 := 0 < r_1 < \cdots < r_N := n$, and where (cf. (1.2))

$$W_j := span\{\mathbf{e}_k : r_{j-1} + 1 \leq k \leq r_j\} \quad (j = 1, 2, \cdots, N).$$

This implies that

$$\dim W_j = r_j - r_{j-1} \quad (j = 1, 2, \cdots, N).$$

As in [7], we remark that the various partitions of $\mathbb{C}^n$ can be partially ordered. If $\pi_1 = \{r_j\}_{j=0}^N$ and $\pi_2 = \{s_j\}_{j=0}^M$ are two partitions of $\mathbb{C}^n$, we write

$$\pi_1 \prec \pi_2 \tag{3.1}$$

if and only if $\{r_j\}_{j=0}^N \subset \{s_j\}_{j=0}^M$, and we say that π_1 is *weaker* than π_2 (and that π_2 is *stronger* than π_1). Evidently, $\pi_s := \{i\}_{i=0}^n$ is the *strongest* partition of $\mathbb{C}^n$, i.e.,

$$\mathbb{C}^n = W_1 \dot{+} W_2 \dot{+} \cdots \dot{+} W_n \text{ and } \dim W_j = 1 \quad (j = 1, 2, \cdots, n),$$

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while $\pi_w := \{0, n\}$ is the *weakest* partition of $\mathbb{C}^n$, i.e.,

$$\mathbb{C}^n = W_1 \text{ and } \dim W_1 = n.$$

The main object of this section is the following result which crudely says that a stronger partition gives a "bigger" minimal Gerschgorin set.

THEOREM 3.1. *Let π_1 and π_2 be two partitions of $\mathbb{C}^n$ with $\pi_1 \prec \pi_2$. Then, for any matrix A in $\mathbb{C}^{n,n}$,*

$$\sigma\left(\Omega_{\pi_1}(A)\right) \subset \sigma\left(\Omega_{\pi_2}(A)\right), \tag{3.2}$$

and

$$G_{\pi_1}(A) \subset G_{\pi_2}(A). \tag{3.3}$$

Proof. If $\pi_1 \prec \pi_2$, it is a direct consequence of the definition in (1.13) that any matrix B in $\Omega_{\pi_1}(A)$ is also in $\Omega_{\pi_2}(A)$, which gives $\Omega_{\pi_1}(A) \subset \Omega_{\pi_2}(A)$. Then, the inclusion of (3.2) is evident. Since $G_{\pi_1}(A)$ is known to be a closed bounded set in $\mathbb{C}$, it suffices, to establish (3.3), to show that $\partial G_{\pi_1}(A) \subset G_{\pi_2}(A)$. For any $z \in \partial G_{\pi_1}(A)$, Theorem 2.1 gives that z is an eigenvalue of some $B \in \Omega_{\pi_1}(A)$. But as $\pi_1 \prec \pi_2, B \in \Omega_{\pi_2}(A)$ also; whence from (1.14), $\partial G_{\pi_1}(A) \subset \sigma\left(\Omega_{\pi_2}(A)\right) \subset G_{\pi_2}(A)$, which gives (3.3). □

The inclusion (3.3) of Theorem 3.1 says that a stronger partition gives a "bigger" minimal Gerschgorin set, but this needs clarification. As was mentioned in [7], if π_1 and π_2 are two partitions of $\mathbb{C}^n$ with

$$\pi_1 \prec \pi_2 \text{ and } \pi_1 \neq \pi_2, \tag{3.4}$$

then from (3.3), $G_{\pi_1}(A) \subset G_{\pi_2}(A)$, but equality in this last inclusion is *not* ruled out. Indeed, an example of this is explicitly given in [7], where (3.4) is valid and where $G_{\pi_1}(A) = G_{\pi_2}(A)$.

To conclude this section, we connect the results of this paper with known results in the literature concerning minimal Gerschgorin sets. First, in the special case that the partition is the strongest partition π_s of $\mathbb{C}^n$ (i.e., $\dim W_j = 1$ for $j = 1, 2, \cdots, n$ in (1.2)), the results of Theorems 2.1 and 2.5 of this paper reduce to the original results of [6]. We mention that the proofs of these results in [6] depended solely on the Perron-Frobenius theory of nonnegative matrices, which sharply differs from the analysis used here. In this regard, we mention that Levinger [5] later obtained a new characterization of the minimal Gerschgorin set in the strongest partition case, which, via the Perron-Frobenius theory, is equivalent to the definition of $H_{\pi_s}(A)$ of (2.6). In this sense, our results in this paper could be viewed more as a generalization of the results of Levinger [5] than of those of [6].

The connection of the results of this paper with those of [7] is certainly very strong. The analogs of Theorems 2.1 and 2.5 do appear in [7], except that two additional hypotheses were needed in [7] in deriving the same results. One of the additional hypotheses, called π-*regularity* in [7], turns out to be a consequence of the new affirmative solution of the Spectral Conjecture in [4]. The other additional hypothesis, called π-*irreducibility* in [7], is not needed because of the new treatment here of the

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spectrum of the associated block diagonal matrices. (This is where Lemmas 2.2 and 2.3 play a role.) We also mention that the set $\hat{\Omega}_\pi(A)$, as defined here in (2.9), is in general a smaller set than that defined in [7], because just *one* τ in $[0,1]$ is associated with each matrix B in $\hat{\Omega}_\pi(A)$ of (2.9), whereas its counterpart in [7] assigns a possibly different τ in $[0,1]$ to *each* non-diagonal submatrix of B in $\hat{\Omega}_\pi(A)$.

4. Examples. For simplicity, consider the 3×3 complex matrix with diagonal elements $1+i, -1+i, -i$ and let its off-diagonal elements be given by the first 6 digits of the decimal expansion of the number $\pi := 2\arctan 1 = 3.14159\ldots$, i.e.,

$$A := \begin{bmatrix} 1+i & 3 & 1 \\ 4 & -1+i & 1 \\ 5 & 9 & -i \end{bmatrix}. \tag{4.1}$$

It can be readily verified that the eigenvalues of A, truncated to four decimal digits, are

$$\sigma(A) = \{5.8129 + 0.4468i, -2.0764 - 0.3197i, -3.7364 + 0.8728i\}. \tag{4.2}$$

Then, all possible partitions of $\mathbb{C}^3$ are given by

$$\begin{aligned} &\pi_1 := \pi_w = \{0,3\}; \\ &\pi_2 := \{0,1,3\}; \\ &\pi_3 := \{0,2,3\}; \\ &\pi_4 := \pi_s = \{0,1,2,3\}. \end{aligned} \tag{4.3}$$

By definition, we have

$$\pi_1 \prec \pi_2 \prec \pi_4, \tag{4.4}$$

and

$$\pi_1 \prec \pi_3 \prec \pi_4, \tag{4.5}$$

which implies from Theorem 3.1 that

$$G_{\pi_1}(A) \subset G_{\pi_2}(A) \subset G_{\pi_4}(A), \tag{4.6}$$

and

$$G_{\pi_1}(A) \subset G_{\pi_3}(A) \subset G_{\pi_4}(A). \tag{4.7}$$

We next explicitly determine the minimal Gerschgorin sets $\left\{G_{\pi_j}(A)\right\}_{j=1}^{4}$ for the matrix A of (4.1). First, because π_1 of (4.1) is the weakest partition of $\mathbb{C}^3$, then $A = D_{\pi_1}$. From this, it follows from (1.11) that $H_{\pi_1}(A) = \emptyset$, and from (1.10) that $G_{\pi_1}(A) = \sigma(D_{\pi_1}) = \sigma(A)$, where $\sigma(A)$ is given in (4.2). Next, we consider π_4, the strongest partition of $\mathbb{C}^3$. Then, its associated block-diagonal matrix is

$$D_{\pi_4} = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -i \end{bmatrix},$$

and for any $B \in \Omega_{\pi_4}(A)$, we have (cf. (1.13)).

$$B = \begin{bmatrix} 1+i & 3e^{i\theta_1} & e^{i\theta_2} \\ 4e^{i\theta_3} & -1+i & e^{i\theta_4} \\ 5e^{i\theta_5} & 9e^{i\theta_6} & -i \end{bmatrix} \quad (\text{any } \{\theta_j\}_{j=1}^{6} \text{ real}), \tag{4.8}$$

so that for any $B \in \Omega_{\pi_4}(A)$,

$$(zI - D_{\pi_4})^{-1}(B - D_{\pi_4}) = \begin{bmatrix} 0 & \frac{3e^{i\theta_1}}{z-1-i} & \frac{e^{i\theta_2}}{z-1-i} \\ \frac{4e^{i\theta_3}}{z+1-i} & 0 & \frac{e^{i\theta_4}}{z+1-i} \\ \frac{5e^{i\theta_5}}{z+i} & \frac{9e^{i\theta_6}}{z+i} & 0 \end{bmatrix} \quad (z \notin \sigma(D_{\pi_4}))\,. \tag{4.9}$$

To determine $\partial H_{\pi_4}(A)$, it suffices from (2.5) and (2.6) to find those z, not in $\sigma(D_{\pi_4})$, for which $\nu(z) = 1$, where (cf. (2.4))

$$\nu(z) := \sup_{B \in \Omega_{\pi_4}(A)} \rho\left\{(zI - D_{\pi_4})^{-1}(B - D_{\pi_4}\right\}. \tag{4.10}$$

From the Perron-Frobenius theory of nonnegative matrices (cf. [8, p. 28]), the choice of the real numbers $\{\theta_i\}_{i=1}^{6}$ in (4.9), which maximizes $\rho\left\{(zI - D_{\pi_4})^{-1}(B - D_{\pi_4})\right\}$ clearly leads to the nonnegative matrix

$$M_{\pi_4}(z) := \begin{bmatrix} 0 & \frac{3}{|z-1-i|} & \frac{1}{|z-1-i|} \\ \frac{4}{|z+1-i|} & 0 & \frac{1}{|z+1-i|} \\ \frac{5}{|z+i|} & \frac{9}{|z+i|} & 0 \end{bmatrix} \quad (z \notin \sigma(D_{\pi_4}))\,. \tag{4.11}$$

whose spectral radius is then the quantity $\nu(z)$ in (4.10). On computing the characteristic polynomial for the matrix $M_{\pi_4}(z)$, it can be verified that $\nu(z) = 1$ if and only if

$$|z-1-i| \cdot |z+1-i| \cdot |z+i| - 9|z-1-i| - 5|z+1-i| - 12|z+i| - 51 = 0. \tag{4.12}$$

Using MATLAB, the set of z's satisfying (4.12) determines the curve $C_4 = \partial H_{\pi_4}(A)$, and this is shown in Figure 4.1. But because the eigenvalues $\{1+i, -1+i, = i\}$ of D_{π_4} all lie in the interior of C_4, it follows (cf. (1.10)) that $C_4 = \partial G_{\pi_4}(A) = \partial H_{\pi_4}(A)$. In a similar way, it can be verified that the set of z's satisfying

$$|z - 1 - i| \cdot |z^2 + z - 8 + i| - |17z + 56 + 7i| = 0, \tag{4.13}$$

determines the curve $C_2 := \partial G_{\pi_2}(A)$ where $\pi_2 = \{0, 1, 3\}$, and that the set of z's, satisfying

$$|z + i| \cdot |z^2 - 2iz - 14| - |14z + 47 - 14i| = 0, \tag{4.14}$$

determines the curve $C_3 := \partial G_{\pi_3}(A)$ where $\pi_3 = \{0, 2, 3\}$. The latter curves are also given in Figure 4.1, along with the three points of the $\sigma(A)$, which are shown as small solid disks. It is interesting to see in Figure 4.1 that the set $G_{\pi_3}(A)$ consists of *two* disjoint compact sets, whose boundaries are denoted by C_{3a} and C_{3b}. Note also from Figure 4.1 that all eigenvalues of A of (4.1) lie on C_2, while two eigenvalues of A lie on C_{3a} and one eigenvalue of A lies on C_{3b}. Moreover, Figure 4.1 directly shows that the set inclusions of (4.6) and (4.7) are clearly valid for the partial orderings in (4.4) and (4.5).

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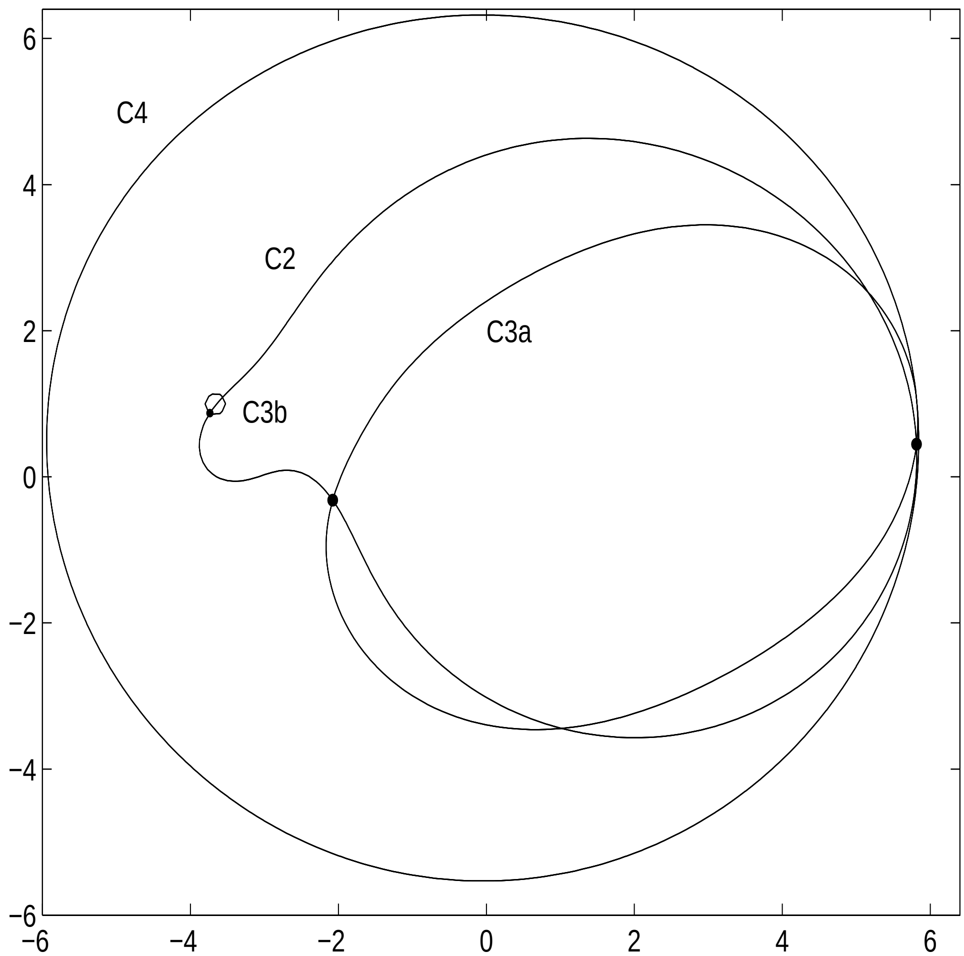

FIG. 4.1. *The curves $C_j := \partial G_{\pi_j}(A)$ for $j = 2, 3, 4$, and $\sigma(A)$.*

Finally, a visual "verification" of (2.12) of Theorem 2.5 is given in Figure 4.2, for the matrix A of (4.1) and for the strongest partition π_4 of $\mathbb{C}^3$ of (4.3). Here, on selecting a finite set of values of τ in $[0, 1]$, and on selecting a finite set of values of each θ_j in $[0, 2\pi]$ for $j = 1, 2, \cdots, 6$, all eigenvalues of each matrix $\tilde{B}$, given by

$$\tilde{B} = \begin{bmatrix} 1+i & 3\tau e^{i\theta_1} & \tau e^{i\theta_2} \\ 4\tau e^{i\theta_3} & -1+i & \tau e^{i\theta_4} \\ 5\tau e^{i\theta_5} & 9\tau e^{i\theta_6} & -i \end{bmatrix} \quad (\text{where } \hat{B} \in \hat{\Omega}_{\pi_4}(A)), \tag{4.15}$$

were determined and plotted in Figure 4.2, showing that the set of all these eigenvalues do indeed tend to "fill out" the minimal Gerschgorin set $G_{\pi_4}(A)$, as established in (2.12) of Theorem 2.5.

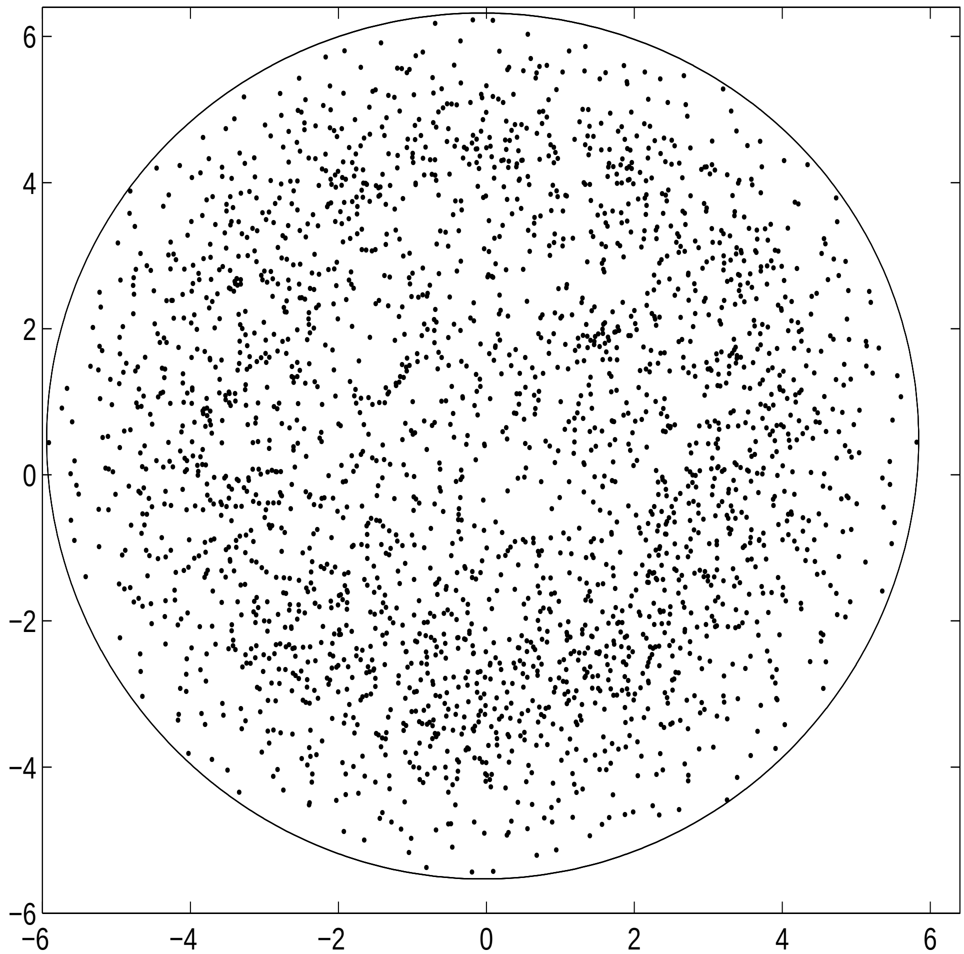

FIG. 4.2. *Points of* $\sigma(\hat{\Omega}_{\pi_4}(A))$ *and* $G_{\pi_4}(A)$.

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