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MINIMIZATION OF THE SPECTRAL NORM OF THE SOR OPERATOR IN A MIXED CASE*

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Dedicated to Gene Golub on the occasion of his 75th birthday

Abstract. In this work we solve the problem of the minimization of the spectral norm of the SOR operator associated with a block two-cyclic consistently ordered matrix $A \in \mathbb{C}^{n,n}$, assuming that the corresponding Jacobi matrix has eigenvalues $\mu \in [-\beta, \beta] \cup [-\imath\alpha, \imath\alpha]$, with $\beta \in [0, 1)$, $\alpha \in [0, +\infty)$ and $\imath = \sqrt{-1}$. Previous results obtained by other researchers are extended.

Key words. Jacobi and SOR iteration matrices, block two-cyclic consistently ordered matrix, spectral matrix norm

AMS subject classifications. 65F10

1. Introduction.

It is known that the k^{th} root of a natural norm of the k^{th} power of the iteration operator of a first order iterative scheme for the solution of a linear system of algebraic equations, with iterative matrix T such that $\rho(T) < 1$, gives a better average convergence measure for k iterations than that of the corresponding spectral radius $\rho(T)$ (see, e.g., [9] or [11]). Having this as a guide and using the Singular Value Decomposition Theorem, Golub and de Pillis [4] recover the well known formulas that connect the eigenvalues of the Jacobi operator with those of the SOR and the Modified SOR ones when the Jacobi iteration matrix is weakly cyclic of index 2 (see, e.g., [9], [11]). Subsequently, they show computationally that the optimization of the relaxation parameter(s) involved based on the minimization of the spectral norms of the associated SOR and MSOR operators after k iterations give better convergence results than those based on the minimization of the spectral radius. Next, Hadjidimos and Neumann [5] solve completely the minimization problem of the spectral norm of the aforementioned two operators, for $k = 1$, under the assumption that the eigenvalue spectrum of the Jacobi iteration matrix B, $\sigma(B)$, is real such that $\sigma(B) \subset [-\beta, \beta]$, $\beta \in [0, 1)$. So, they extend and complete the work started by Young and his colleagues (see [11] and the related references cited therein). In a recent work, Milléo, Yin and Yuan [8] solve the corresponding problems when the spectrum of the Jacobi matrix is purely imaginary satisfying $\sigma(B) \subset [-\imath\alpha, \imath\alpha]$, $\alpha \in [0, +\infty)$, with $\imath = \sqrt{-1}$. In this work we extend the previous results and solve the problem of minimization of the spectral norm of the SOR operator in the mixed case; that is in the case where the Jacobi matrix has both real and purely imaginary eigenvalues satisfying $\sigma(B) \subset [-\beta, \beta] \cup [-\imath\alpha, \imath\alpha]$, $\beta \in [0, 1)$, $\alpha \in [0, +\infty)$. As will be seen, the analysis is not so trivial because a tremendous number of cases have to be examined. The conclusion we end up with is that the value of the optimal SOR parameter $\widehat{\omega}$ is given by three different expressions that depend on the ordering of certain values of functions of β and α.

Specifically, the optimal results obtained are given by the following theorem and the associated Table 1.1.

THEOREM 1.1. *Let $A \in \mathbb{C}^{n,n}$ be block two-cyclic consistently ordered with associated Jacobi iteration matrix B whose spectrum is given by $\sigma(B) \subset [-\beta, \beta] \cup [-\imath\alpha, \imath\alpha]$, $\beta \in [0, 1)$, $\alpha \in [0, +\infty)$, with $\imath$ denoting the imaginary unit. Then, the value of the optimal*

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TABLE 1.1
Minimum value of $S(d^2,\omega) := \|\mathcal{L}_\omega\|_2$. Notation: The function $S(d^2,\omega)$ is given by means of (2.14), (2.15) and (2.16); $\widehat{\omega}_1$, $\widehat{\omega}_2$ and ω_2 are given by Theorem 3.1, by (3.4) of Theorem 3.2, and by (3.12), respectively; α_0 and α_1 are given in (4.2) and (4.7).

Relative position of β and α	Restrictions on β and α	$\widehat{\omega}$	$\min S(d^2,\omega)$
$\beta=\alpha=0$		1	0
$0<\beta=\alpha<1$		$\widehat{\omega}_1$	$S(\beta^2,\widehat{\omega}_1)$
$0=\beta<\alpha$		$\widehat{\omega}_2$	$S(\alpha^2,\widehat{\omega}_2)$
$0<\beta$ $<\min\{\alpha,1\}$	$0<\beta<\sqrt{\frac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}$ $\alpha_0\le\alpha\le\alpha_1$	$\widehat{\omega}_2$	$S(\alpha^2,\widehat{\omega}_2)$
	$\sqrt{\frac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}<\beta<\min\{\alpha,1\}$ $\alpha_0\le\alpha\le\alpha_1$	ω_2	$S(\beta^2,\omega_2)$ $\equiv S(\alpha^2,\omega_2)$
	$\sqrt{\frac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}<\beta<\min\{\alpha,1\}$ $\alpha>\alpha_1$	$\widehat{\omega}_1$	$S(\beta^2,\widehat{\omega}_1)$
$0\le\alpha<\beta<1$		$\widehat{\omega}_1$	$S(\beta^2,\widehat{\omega}_1)$

(minimum) spectral norm of the SOR iteration matrix, $\|\mathcal{L}_\omega\|_2$, is given in Table 1.1 by means of the values of the function $S(d^2,\widehat{\omega})$, where d is either β or α, and $\widehat{\omega}$ is the optimal relaxation parameter.

In section 2, some preliminary notation, terminology and statements are presented. Also, following the main ideas in [4] and [5] we are led to the statement of the minimization problem of $\|\mathcal{L}_\omega\|_2$. It is found that its solution depends on the relative position of β and α and also on that of two functions of ω, $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ given there. In section 3, the relative position of $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ is determined. In section 4, the ordering of the possible optimal values for ω, that is of $\widehat{\omega}_1$, $\widehat{\omega}_2$ and ω_2, is established. In section 5, the determination of the optimal ω, $\widehat{\omega}$, is accomplished. Finally, in section 6, we find the position of $\widehat{\omega}$ of this work with respect to other optimal ω's that have been found under the same assumptions on $\sigma(B)$, where an average asymptotic convergence factor was minimized instead of $\|\mathcal{L}_\omega\|_2$.

2. Preliminaries and background material. Suppose that the coefficient matrix $A\in\mathbb{C}^{n,n}$ of a linear system of algebraic equations is block two-cyclic consistently ordered. Then, without loss of generality, we may assume that A has the form

$$A = \begin{bmatrix} I_p & -M \\ -N & I_q \end{bmatrix}, \tag{2.1}$$

where $p+q=n$ and $p\ge q$. According to the Singular Value Decomposition (SVD) Theorem (see, e.g., [3] or [7]), for $M\in\mathbb{C}^{p,q}$ there exist unitary matrices $U\in\mathbb{C}^{p,p}$, $V\in\mathbb{C}^{q,q}$ and a real nonnegative "diagonal" matrix $\Sigma\in\mathbb{C}^{p,q}$, with "diagonal" elements $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_q\ge 0$, such that

$$M = U\Sigma V^H. \tag{2.2}$$

Suppose that N in (2.1) is connected with M via the relation

$$N = VEV^H M^H, \quad \text{whence } N = V(E\Sigma^T)U^H, \tag{2.3}$$

with E being a diagonal matrix given by

$$E \;=\; \mathrm{diag}(e_1,\, e_2,\, \ldots,\, e_q), \tag{2.4}$$

where each of the e_i's has either the value 1 or -1. The above choice for N is justified from the following: a) If all e_i's are equal to 1 then A is Hermitian and the Jacobi iteration matrix associated with it, that is

$$B = \begin{bmatrix} O_{pp} & M \\ N & O_{qq} \end{bmatrix}, \tag{2.5}$$

has all its eigenvalues μ real with $\mu^2 \geq 0$. b) If all e_i's are equal to -1 then B is skew-Hermitian and has all its eigenvalues μ purely imaginary with $\mu^2 \leq 0$. c) If some of the e_i's have the value 1 and the rest the value -1, then some of the μ's are real, the rest are purely imaginary, and then $\mu^2 \in \mathbb{R}$.

From (2.5) we have $B^2 = \begin{bmatrix} MN & O_{pq} \\ O_{qp} & NM \end{bmatrix}$. Therefore, for $\mu \in \sigma(B)$ it is $\mu^2 \in \sigma(B^2)$, and so $\mu^2 \in \sigma(MN) \cup \sigma(NM)$. From (2.2) and (2.3) we have $MN = U\Sigma E\Sigma^T U^H$ and $NM = VE\Sigma^T\Sigma V^H$. Consequently, $\sigma(MN) = \sigma(\Sigma E\Sigma^T)$ and $\sigma(NM) = \sigma(E\Sigma^T\Sigma)$, where

$$\Sigma E\Sigma^T = \begin{bmatrix} e_1\sigma_1^2 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & e_2\sigma_2^2 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \ldots & e_q\sigma_q^2 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{p,p},$$

$$E\Sigma^T\Sigma = \begin{bmatrix} e_1\sigma_1^2 & 0 & \ldots & 0 \\ 0 & e_2\sigma_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e_q\sigma_q^2 \end{bmatrix} \in \mathbb{R}^{q,q}.$$

In view of the form of A in (2.1), the associated SOR iteration matrix $\mathcal{L}_\omega$ will be

$$\mathcal{L}_\omega \;=\; \begin{bmatrix} (1-\omega)I_p & \omega M \\ \omega(1-\omega) & \omega^2 NM + (1-\omega)I_q \end{bmatrix},$$

where $\omega \in (0, 2)$ is the relaxation parameter. Using the expressions in (2.2) and (2.3) for M and N we have

$$\mathcal{L}_\omega \;=\; \begin{bmatrix} U & O \\ O & V \end{bmatrix} \begin{bmatrix} (1-\omega)I_p & \omega\Sigma \\ \omega(1-\omega)E\Sigma^T & \omega^2 E\Sigma^T\Sigma + (1-\omega)I_q \end{bmatrix} \begin{bmatrix} U^H & O \\ O & V^H \end{bmatrix}.$$

Setting $Q := \begin{bmatrix} U & O \\ O & V \end{bmatrix}$ and using an appropriate permutation matrix P, see [4], we can express $\mathcal{L}_\omega$ in the form

$$\mathcal{L}_\omega \;=\; (QP^T)\Delta_\omega(PQ^H), \tag{2.6}$$

with

$$\Delta(\omega) = \begin{bmatrix} \Delta_1(\omega) & O & \cdots & O & O \\ O & \Delta_2(\omega) & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & \Delta_q(\omega) & O \\ O & O & \cdots & O & (1-\omega)I_{p-q} \end{bmatrix} \tag{2.7}$$

and

$$\Delta_i(\omega) = \begin{bmatrix} 1-\omega & \omega\sigma_i \\ \omega(1-\omega)e_i\sigma_i & \omega^2 e_i\sigma_i^2 + 1 - \omega \end{bmatrix} \in \mathbb{R}^{2,2},\ i = 1(1)q. \tag{2.8}$$

From (2.6), (2.7) and because $(QP^T)^{-1} = PQ^H$, we readily obtain

$$\|\mathcal{L}_\omega\|_2^2 = \rho\left(\mathcal{L}_\omega^H \mathcal{L}_\omega\right) = \rho\left(\Delta^H(\omega)\Delta(\omega)\right) = \|\Delta(\omega)\|_2^2.$$

Then, from (2.6)–(2.8), one takes

$$\|\mathcal{L}_\omega\|_2^2 = \max\left\{\max_{i=1(1)q} \|\Delta_i(\omega)\|_2^2,\ (1-\omega)^2\right\}, \tag{2.9}$$

where

$$\Delta_i^H(\omega)\Delta_i(\omega) = \begin{bmatrix} (1-\omega)^2(1+\omega^2\sigma_i^2) & \omega\sigma_i(1-\omega)[1+(1-\omega)e_i+\omega^2\sigma_i^2] \\ \omega\sigma_i(1-\omega)[1+(1-\omega)e_i+\omega^2\sigma_i^2] & \omega^2\sigma_i^2 + (1-\omega+\omega^2 e_i\sigma_i^2)^2 \end{bmatrix}. \tag{2.10}$$

Note that if any $\sigma_i = 0$, then $\|\Delta_i(\omega)\|_2^2 = (1-\omega)^2$. Hence, in view of (2.10), (2.9) is simplified to

$$\|\mathcal{L}_\omega\|_2^2 = \max_{i=1(1)q} \|\Delta_i(\omega)\|_2^2 = \max_{i=1(1)q} \frac{1}{2}\left[T_i + \sqrt{T_i^2 - 4c}\right], \tag{2.11}$$

where

$$\begin{aligned} T_i &:= T(\omega, e_i, \sigma_i^2) = (1-\omega)^2(1+\omega^2\sigma_i^2) + \omega^2\sigma_i^2 + (1-\omega+\omega^2 e_i\sigma_i^2)^2 \geq 0, \\ c &:= c(\omega) = (1-\omega)^4 \geq 0. \end{aligned} \tag{2.12}$$

Note that in our analysis we assumed that $q \leq p$. If the last inequality is reversed then the maximum in (2.11) will be taken for all $i = 1(1)p$. So, formula (2.9) would cover both cases if it was given in the following form,

$$\|\mathcal{L}_\omega\|_2^2 = \max_{i=1(1)\min\{p,q\}} \frac{1}{2}\left[T_i + \sqrt{T_i^2 - 4c}\right]. \tag{2.13}$$

Now, we can observe that $\frac{d}{dT_i}\|\mathcal{L}_\omega\|_2^2 = \frac{1}{2}\left[1 + \frac{T_i}{\sqrt{T_i^2-4c}}\right] > 0$. Also,

$$\frac{\partial T_i}{\partial(\sigma_i^2)} = \omega^2[(1-\omega+e_i)^2 + 2\omega^2\sigma_i^2] > 0 \quad \text{and} \quad \frac{\partial^2 T_i}{\partial(\sigma_i^2)^2} = 2\omega^4 > 0.$$

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Hence, T_i is an increasing and convex function of σ_i^2. Considering (2.12), we can see that each T_i is given by one of two different expressions depending on the value of e_i. Hence, we end up with the conclusion that $\|\mathcal{L}_\omega\|_2^2$ in (2.13) will be given by

(2.14)
$$\|\mathcal{L}_\omega\|_2^2 = \begin{cases} \frac{1}{2}\left[T(\beta^2,\omega) + \sqrt{T^2(\beta^2,\omega) - 4(1-\omega)^4}\right], & \text{if } T(\beta^2,\omega) \geq T(\alpha^2,\omega), \\ \frac{1}{2}\left[T(\alpha^2,\omega) + \sqrt{T^2(\alpha^2,\omega) - 4(1-\omega)^4}\right], & \text{if } T(\beta^2,\omega) \leq T(\alpha^2,\omega), \end{cases}$$

where

$$\begin{aligned} T(\beta^2,\omega) &= (1-\omega)^2(1+\omega^2\beta^2) + \omega^2\beta^2 + (1-\omega+\omega^2\beta^2)^2, \\ T(\alpha^2,\omega) &= (1-\omega)^2(1+\omega^2\alpha^2) + \omega^2\alpha^2 + (1-\omega-\omega^2\alpha^2)^2. \end{aligned} \tag{2.15}$$

Let us introduce the function

$$S(d^2,\omega) := \|\mathcal{L}_\omega\|_2, \tag{2.16}$$

where d is either β or α, whichever applies from (2.14). The function $S(d^2,\omega)$ will be very useful in the end of our analysis in order to give the optimal results in a compact form.

So, the main problem of this work is the following:

Problem: Find the optimal value of ω, $\widehat{\omega}$, that minimizes $S(d^2,\omega)$ in (2.16) or, equivalently, $\|\mathcal{L}_\omega\|_2^2$ in (2.14).

The solution to our problem is achieved in three steps: i) Consider only the values of $\omega \in (0,2)$ for which the SOR converges. ii) Work with the considered ω's, determine which of the two expressions $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ is the largest. iii) Minimize the largest of the two expressions as a function of ω.

3. Ordering $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ as functions of ω.

Before we go on with our analysis the reader is reminded of the following: For a weakly cyclic of index 2 Jacobi matrix, the optimal values of ω, $\widehat{\omega}$, that minimize $\|\mathcal{L}_\omega\|_2^2$ in (2.14), for $\rho(\mathcal{L}_\omega) < 1$, when a) $\mu^2 \in [0,\beta^2]$, $\beta \in [0,1)$ and b) $\mu^2 \in [-\alpha^2, 0]$, $\alpha \in [0,+\infty)$, were obtained in [5] and [8], respectively. They are given by the following statements.

a) $\underline{\sigma(B^2) \subset [0,\beta^2],\ \beta \in [0,1)}$:

THEOREM 3.1. *The value of $\widehat{\omega} = \widehat{\omega}_1 \in (0,2)$ that minimizes $\|\mathcal{L}_\omega\|_2$ is the unique positive real root in $(0,1)$ of the quartic equation*

$$\begin{aligned} f(\omega) := {} & (\beta^4+\beta^6)\omega^4 + (1-4\beta^4)\omega^3 + (-5+4\beta^2+4\beta^4)\omega^2 \\ & +(8-8\beta^2)\omega + (-4+4\beta^2) \\ = {} & 0. \end{aligned} \tag{3.1}$$

More specifically, $\widehat{\omega}_1 \in (0,\omega^*)$, where ω^* is the unique positive real root in $(0,1)$ of the cubic equation

$$g(\omega) := (\beta^2+\beta^4)\omega^3 - 3\beta^2\omega^2 + (1+2\beta^2)\omega - 1 = 0. \tag{3.2}$$

Moreover, $\|\mathcal{L}_\omega\|_2$, as a function of ω, strictly decreases in $[0,\widehat{\omega}_1]$ and strictly increases in $[\widehat{\omega}_1, 1]$.

b) $\underline{\sigma(B^2) \subset [-\alpha^2,0],\ \alpha \in [0,+\infty)}$:

THEOREM 3.2. *The value of $\widehat{\omega} = \widehat{\omega}_2 \in (0, \frac{2}{1+\alpha})$ that minimizes $\|\mathcal{L}_\omega\|_2$ is the unique positive real root in $(0,1)$ of the quadratic*

$$h(\omega) := \alpha^2(\alpha^2+1)\omega^2 + \omega - 1 = 0, \tag{3.3}$$

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given by

$$\widehat{\omega} \;=\; \widehat{\omega}_2 \;=\; \frac{1}{1+\alpha^2}. \tag{3.4}$$

Moreover, $\|\mathcal{L}_\omega\|_2$*, as a function of* ω*, strictly decreases in* $[0, \widehat{\omega}_2]$ *and strictly increases in* $[\widehat{\omega}_2, 1]$.

Note that we are interested in a case where $\alpha\beta > 0$, since for $\beta = 0$ or $\alpha = 0$ we are in one of the cases of Theorems 3.1 or 3.2 and not in a "mixed case" as we would like. To find the real interval of ω for which the SOR method converges when the eigenvalues of the Jacobi iteration matrix B are such that $\mu \in \sigma(B) \subset [-\beta, \beta] \cup [-\imath\alpha, \imath\alpha]$, or, equivalently, $-\alpha^2 \le \mu^2 \le \beta^2$, we begin with Young's famous equation

$$(\lambda + \omega - 1)^2 \;=\; \mu^2\omega^2\lambda, \tag{3.5}$$

with $\lambda \in \sigma(\mathcal{L}_\omega)$. To find the ω's for which $|\lambda| < 1$ we use Lemma 6.2.1 of [11], which gives the conditions of the Schur-Cohn algorithm for a polynomial to have all its zeros strictly less than 1 in modulus [6]. In our case we have

$$p(\lambda) \;:=\; \lambda^2 - [\mu^2\omega^2 - 2(\omega-1)]\lambda + (\omega-1)^2 = 0, \tag{3.6}$$

and the Schur-Cohn conditions are

$$|(\omega-1)^2| < 1 \;\text{ and }\; |\mu^2\omega^2 - 2(\omega-1)| < 1 + (\omega-1)^2. \tag{3.7}$$

The first condition gives that $\omega \in (0, 2)$, the well known Kahan's necessary condition for the SOR to converge. The second condition, using the fact that $-\alpha^2 \le \mu^2 \le \beta^2$ and distinguishing the three cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$, gives $\beta \in [0, 1)$ and $0 < \omega < \frac{2}{1+\alpha}$.

As was mentioned in section 2, we must determine the largest of $T(\beta^2, \omega)$ and $T(\alpha^2, \omega)$ in (2.14). Thus, we form the difference $T(\beta^2, \omega) - T(\alpha^2, \omega)$ and find

$$\begin{aligned} T(\beta^2,\omega) - T(\alpha^2,\omega) \;&=\; \omega^2\left[(\beta^2-\alpha^2)(1+\beta^2+\alpha^2)\omega^2 - 4\beta^2\omega + 4\beta^2\right] \\ &=: \omega^2 P(\omega). \end{aligned} \tag{3.8}$$

In case $\beta = \alpha$ we have that $P(\omega)_{|\beta=\alpha} = 4\beta^2(1-\omega)$, meaning that $T(\beta^2, \omega) \ge T(\alpha^2, \omega)\,, \forall \omega \in (0, 1]$, while $T(\beta^2, \omega) \le T(\alpha^2, \omega)\,, \forall \omega \in [1, \frac{2}{1+\alpha})$.

In the sequel we examine the main case $\beta \ne \alpha$. To consider and study simplified expressions we introduce the notation

$$A \;\sim\; B, \tag{3.9}$$

meaning that the two expressions or quantities A and B are of the same sign.

First, we distinguish two cases according to the sign of the discriminant of the quadratic $P(\omega)$ in (3.8), since

$$T(\beta^2,\omega) - T(\alpha^2,\omega) \;\sim\; P(\omega). \tag{3.10}$$

We have

$$D \;:=\; 16\beta^2\left[\sqrt{\alpha^4+\alpha^2} - \beta^2\right]\left[\sqrt{\alpha^4+\alpha^2} + \beta^2\right] \;\sim\; \sqrt[4]{\alpha^4+\alpha^2} - \beta. \tag{3.11}$$

Case I) $D \le 0$. This is equivalent to $\sqrt[4]{\alpha^4+\alpha^2} \le \beta\,(< 1)$. Note, for such a case to exist, $0 < \alpha < \sqrt{\frac{-1+\sqrt{5}}{2}}$ must hold. Since $D \le 0$, it is

$$P(\omega) \sim (\beta^2-\alpha^2)(1+\beta^2+\alpha^2) \sim \beta - \alpha,$$

because the quadratic shares the same sign with the coefficient of its leading term in this case. However, it is $\alpha < \sqrt[4]{\alpha^4+\alpha^2} \le \beta$, and so $P(\omega) > 0$, implying that

$$T(\beta^2,\omega) > T(\alpha^2,\omega)\,, \quad \forall\omega \in \left(0, \frac{2}{1+\alpha}\right).$$

Case II) $D > 0$. This is equivalent to $\beta < \sqrt[4]{\alpha^4+\alpha^2}$ or, to be more specific, to $\beta < \min\{1, \sqrt[4]{\alpha^4+\alpha^2}\}$. $P(\omega)$ in (3.8) has two real zeros given by

$$\begin{aligned}\omega_1 &= \frac{2\beta(\beta+\sqrt{\alpha^4+\alpha^2-\beta^4})}{(\beta^2-\alpha^2)(1+\alpha^2+\beta^2)},\\ \omega_2 &= \frac{2\beta(\beta-\sqrt{\alpha^4+\alpha^2-\beta^4})}{(\beta^2-\alpha^2)(1+\alpha^2+\beta^2)} = \frac{2\beta}{\beta+\sqrt{\alpha^4+\alpha^2-\beta^4}}.\end{aligned} \tag{3.12}$$

Because of the difference $\beta^2-\alpha^2$ in the denominator of the fraction giving ω_1, two subcases are to be considered, depending on the ordering of β and α.

Subcase IIa) $\alpha < \beta < \min\{1, \sqrt[4]{\alpha^4+\alpha^2}\}$. It is readily seen that $0 < \omega_2 < \omega_1$. Also,

$$\omega_1 - \frac{2}{1+\alpha} \sim \sqrt{\alpha^4+\alpha^2-\beta^4}[(1+\alpha)\beta + \sqrt{\alpha^4+\alpha^2-\beta^4}] + \alpha\beta^2 > 0.$$

On the other hand we have $P(\frac{2}{1+\alpha}) = \frac{4}{(1+\alpha)^2}(b^4+\alpha^2\beta^2-\alpha^4-\alpha^2)$. The quantity in the last pair of parentheses is nonnegative if and only if $\beta \ge \sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}$. However, the last expression is always greater than α, since $\alpha < 1$ for $\alpha < \beta < 1$, and strictly less than $\sqrt[4]{\alpha^4+\alpha^2}$, as is easily proved. So the sign of $P(\frac{2}{1+\alpha})$ depends on the relative position of β and $\sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}$. Therefore, two further subcases must be distinguished.

Subcase IIa1) $\alpha < \sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}} \le \beta < \min\left\{1, \sqrt[4]{\alpha^4+\alpha^2}\right\}$. From the previous analysis we have that in this case it is $P(\frac{2}{1+\alpha}) \ge 0$, implying that $\omega_2 \ge \frac{2}{1+\alpha}$. Hence $T(\beta^2,\omega) > T(\alpha^2,\omega)$, $\forall\omega \in \left(0, \frac{2}{1+\alpha}\right)$.

Subcase IIa2) $\alpha < \beta < \sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}$ (< 1). In this case $P(\frac{2}{1+\alpha}) < 0$, therefore $\omega_2 < \frac{2}{1+\alpha}$, implying that $T(\beta^2,\omega) \ge T(\alpha^2,\omega)$, $\forall\omega \in (0,\omega_2]$, while $T(\beta^2,\omega) \le T(\alpha^2,\omega)\, \forall\omega \in [\omega_2, \frac{2}{1+\alpha})$.

Subcase IIb) $\beta < \min\{\alpha, 1\}$. It is readily seen that $\omega_1 < 0$ while ω_2 is always positive. Also, ω_2 is strictly less than $\frac{2}{1+\alpha}$ because $P(\frac{2}{1+\alpha}) \sim (\beta^4-\alpha^4) + \alpha^2(\beta^2-1) < 0$, since β is strictly less than both α and 1. Consequently, $T(\beta^2,\omega) \ge T(\alpha^2,\omega)$, $\forall\omega \in (0,\omega_2]$, while $T(\beta^2,\omega) \le T(\alpha^2,\omega)$, $\forall\omega \in [\omega_2, \frac{2}{1+\alpha})$.

Note, fixing α (< 1), we can see that $1-\omega_2 \sim \sqrt{\alpha^4+\alpha^2-\beta^4} - \beta < 0$, so $\omega_2 > 1$. Therefore, for $\beta \to \alpha^+$ (**Subcase IIa2**) we have $\omega_2 \to 1^+$ and $\omega_1 \to +\infty$, while for $\beta \to \alpha^-$ (**Subcase IIb**), $\omega_2 \to 1^-$ and $\omega_1 \to -\infty$. From these observations it is concluded that the case $\beta = \alpha$ can be incorporated in either of the aforementioned subcases, preferably in **Subcase IIa2**.

Having this in mind, and recalling from **Subcase IIa** that $\alpha - 1 \sim \alpha - \sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}$,

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we can summarize all the results so far as follows:

$$
\begin{aligned}
&\text{if } \sqrt[4]{\alpha^4+\alpha^2} \le \beta < 1,\\
&\qquad \text{then } T(\beta^2,\omega) > T(\alpha^2,\omega),\ \forall \omega \in (0, \tfrac{2}{1+\alpha}),\\
&\text{if } \sqrt{\tfrac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}} \le \beta < \min\left\{1, \sqrt[4]{\alpha^4+\alpha^2}\right\},\\
&\qquad \text{then } T(\beta^2,\omega) > T(\alpha^2,\omega),\ \forall \omega \in (0, \tfrac{2}{1+\alpha}),\\
&\text{if } \alpha \le \beta < \sqrt{\tfrac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}},\\
&\qquad \text{then } \begin{cases} T(\beta^2,\omega) \ge T(\alpha^2,\omega), & \text{for } \omega \in (0,\omega_2],\\ T(\beta^2,\omega) \le T(\alpha^2,\omega), & \text{for } \omega \in [\omega_2, \tfrac{2}{1+\alpha}), \end{cases}\\
&\text{if } 0 < \beta < \min\{\alpha, 1\},\\
&\qquad \text{then } \begin{cases} T(\beta^2,\omega) \ge T(\alpha^2,\omega), & \text{for } \omega \in (0,\omega_2],\\ T(\beta^2,\omega) \le T(\alpha^2,\omega), & \text{for } \omega \in [\omega_2, \tfrac{2}{1+\alpha}). \end{cases}
\end{aligned}
\tag{3.13}
$$

4. Ordering the Values of $\widehat{\omega}_1$, $\widehat{\omega}_2$ and ω_2. Having ordered, in (3.13), $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ for all values of $\omega \in (0, \frac{2}{1+\alpha})$ for which the SOR converges, we come now to order $\widehat{\omega}_1$, $\widehat{\omega}_2$ and ω_2. This new ordering together with that of $T(\beta^2,\omega)$ and $T(\alpha^2,\omega)$ will enable us to decide which of the possible optimal ω's gives $\widehat{\omega}$.

First, from the various cases examined in section 3 and especially from the summary in (3.13), it is seen that the only immediate result regarding $\widehat{\omega}$ can be found in the union of **Case I** and **Subcase IIa1**; namely, for $\beta \in \left[\sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}, 1\right)$ and $\alpha < 1$. Then, $T(\beta^2,\omega) > T(\alpha^2,\omega)$, $\forall \omega \in (0, \frac{2}{1+\alpha})$, and $(0,1) \subset (0, \frac{2}{1+\alpha})$. Hence, by virtue of Theorem 3.1, $\widehat{\omega} = \widehat{\omega}_1$.

Next, in **Subcase IIa2** where $\beta \in \left[\alpha, \sqrt{\frac{-\alpha^2+\sqrt{5\alpha^4+4\alpha^2}}{2}}\right)$ and $\alpha < 1$, it was found in the Note in the end of section 3 that $\omega_2 \ge 1$ implying that $T(\beta^2,\omega) \ge T(\alpha^2,\omega)$. Therefore, we have again by Theorem 3.1 that $\widehat{\omega} = \widehat{\omega}_1$.

Finally, in **Subcase IIb** it is $\omega_2 < 1$ and so the relative position of ω_2 with respect to the other two possible optima $\widehat{\omega}_1$, $\widehat{\omega}_2$ has to be determined.

First, we determine the relative position of $\widehat{\omega}_1$ and $\widehat{\omega}_2$. For this we form $f(\widehat{\omega}_2)$, where f is the function defining $\widehat{\omega}_1$ in (3.1). Using Maple 9 we solve the equation $f(\widehat{\omega}_2) = 0$, considering α as the unknown, and find the eight roots given below:

$$
\alpha_{1,2,\ldots,8} = \pm\frac{1}{2}\sqrt{-2 \pm \sqrt{\frac{2\left(1-2\beta^2+4\beta^4 \pm \sqrt{1+8\beta^4}\right)}{1-\beta^2}}}. \tag{4.1}
$$

Four of the eight roots are complex numbers. Specifically, the ones with the minuses in front of the first inner square root. Also, two more roots are complex; the ones with the plus sign in front of the first inner square root and the minus sign in front of the second inner square root. This is because

$$
\begin{aligned}
(1-2\beta^2+4\beta^4) - \sqrt{1+8\beta^4} &\sim (1-2\beta^2+4\beta^4)^2 - (1+8\beta^4)\\
&= 4\beta^2(\beta^2-1)(1+4\beta^4) < 0.
\end{aligned}
$$

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Of the two real opposite in sign roots the positive one is

$$\alpha_0 \;=\; \frac{1}{2}\sqrt{-2+\sqrt{\frac{2\left(1-2\beta^2+4\beta^4+\sqrt{1+8\beta^4}\right)}{1-\beta^2}}}. \tag{4.2}$$

Since the leading coefficient and the constant term of $(1+\alpha^2)^4 f(\frac{1}{1+\alpha^2})$ are positive multiples of $-4+4\beta^2<0$ and $\beta^4+\beta^6>0$, respectively, we have that

$$\mathcal{A}:\ \widehat{\omega}_1 \le \widehat{\omega}_2 \ \text{ for }\ (\beta <)\,\alpha \le \alpha_0 \quad\text{and}\quad \mathcal{B}:\ \widehat{\omega}_1 \ge \widehat{\omega}_2 \ \text{ for }\ \alpha \ge \alpha_0\,(,\beta). \tag{4.3}$$

It can be found that the function of β^2 under the second inner square root in (4.2) has a positive derivative in $[0,1)$. Hence, α_0 is a strictly increasing function of $\beta\in[0,1)$ taking all the values in $[0,+\infty)$. Moreover, it is $\alpha_0\le\beta$ for $\beta\in[0,0.81942756935)$, $\alpha_0>\beta$ for $\beta\in(0.81942756935,1)$.

To find the relative position of ω_2 with respect to $\widehat{\omega}_1$, we find the sign of $f(\omega_2)$. Thus we have, after a number of operations using Maple 9, that

$$f(\omega_2) \sim\ f_1(\alpha)f_3(\alpha), \tag{4.4}$$

where

(4.5)

$$\begin{aligned} f_1(\alpha) &= (1-\beta^2)\alpha^8+(2-2\beta^2)\alpha^6+(1-2\beta^2-2\beta^6)\alpha^4-(\beta^2+2\beta^6)\alpha^2-(\beta^8+\beta^{10}),\\ f_2(\alpha) &= \alpha^8+2\alpha^6+(1+6\beta^2-2\beta^4)\alpha^4+(6\beta^2-2\beta^4)\alpha^2+\beta^4-6\beta^6+\beta^8,\\ f_3(\alpha) &= f_2(\alpha)-4\beta(\alpha^2+1-\beta^2)(\alpha^2+\beta^2)\sqrt{\alpha^2+\alpha^4-\beta^4}. \end{aligned}$$

The polynomial $f_1(\alpha)$ in (4.5) has eight zeros, which are found by Maple 9, namely

$$\alpha_{1,2,\ldots,8} \;=\; \pm\frac{1}{2}\sqrt{-2\pm 2\sqrt{\frac{1+\beta^2+4\beta^6\pm 2\sqrt{\beta^4+8\beta^8}}{1-\beta^2}}}. \tag{4.6}$$

Four of the eight zeros in (4.6), namely the ones with minus in front of the first inner square root, are complex numbers. Also, even if $1+\beta^2+4\beta^6-2\sqrt{\beta^4+8\beta^8}>0$, it is

$$\begin{aligned} -2+2\sqrt{\frac{1+\beta^2+4\beta^6-2\sqrt{\beta^4+8\beta^8}}{1-\beta^2}} &\sim -1+\frac{1+\beta^2+4\beta^6-2\sqrt{\beta^4+8\beta^8}}{1-\beta^2}\\ &\sim 1+2\beta^4-\sqrt{1+8\beta^4}\ \sim\ \beta^4-1<0, \end{aligned}$$

implying that two more zeros, the ones with the minus sign in front of the second inner square root, are complex numbers. So, only two of them are real and opposite in sign. (The fact that $f_1(\alpha)$ in (4.5) has only one positive root as a quartic polynomial in α^2 can be directly checked by Descartes' rule of signs.) Note that the first two coefficients of $f_1(\alpha)$ are positive and the last two negative. So, there is only one change in sign no matter what the sign of the coefficient of the middle term is.) Comparing the positive zero

$$\alpha_1 \;=\; \frac{1}{2}\sqrt{-2+2\sqrt{\frac{1+\beta^2+4\beta^6+2\sqrt{\beta^4+8\beta^8}}{1-\beta^2}}} \tag{4.7}$$

against β it is found that $\alpha_1>\beta$ for all $\beta\in(0,1)$. As in the case of α_0, the derivative of the function of β^2 under the second inner square root is positive in $[0,1)$. Therefore, α_1 as

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a function of $\beta \in [0, 1)$, is a strictly increasing one and can take all the values in $[0, +\infty)$. Consequently, the sign of the polynomial $f_1(\alpha)$, in (4.5) in **Subcase IIb** we are examining, is given below,

$$f_1(\alpha) \begin{cases} \leq 0 \text{ if } \alpha \in (\beta, \alpha_1], \\ \geq 0 \text{ if } \alpha \in [\alpha_1, +\infty). \end{cases} \tag{4.8}$$

Note that comparing α_0 in (4.2) against α_1 in (4.7), it can be found out that $\alpha_0 < \alpha_1$, $\forall \beta \in (0, 1)$.

The polynomial $f_2(\alpha)$ in (4.5) as a function of α has eight zeros which are found by Maple 9 to be

$$\alpha_{1,2,\ldots,8} = \pm\frac{1}{2}\sqrt{-2 \pm 2\sqrt{1 - 12\beta^2 \pm 8\sqrt{2}\beta^2 + 4\beta^4}}. \tag{4.9}$$

Checking the signs of the coefficients of the quartic polynomial in α^2 in (4.5) it is seen that all of them with the possible exception of the constant term are positive. So, if the constant term is positive, which happens if and only if $\beta \in \left(0, \sqrt{3 - 2\sqrt{2}}\right)$, then none of the four zeros α^2 of $f_2(\alpha)$ is positive and then $f_2(\alpha) > 0$. If $\beta \in \left(\sqrt{3 - 2\sqrt{2}}, 1\right)$, then the constant term is negative, one of the zeros α^2 is positive, hence $f_2(\alpha)$ has two real zeros with the positive one being given by

$$\alpha_2 = \frac{1}{2}\sqrt{-2 + 2\sqrt{1 - 12\beta^2 + 8\sqrt{2}\beta^2 + 4\beta^4}}. \tag{4.10}$$

However, since we are interested in a case where $\alpha > \beta$, we compare α_2 above against β. It is found that $\beta - \alpha_2 \sim (1 + 2\beta^2)^2 - (1 - 12\beta^2 + 8\sqrt{2}\beta^2 + 4\beta^4) \sim (2 - \sqrt{2})\beta^2 > 0$. Therefore, $\alpha > \alpha_2$, implying that $f_2(\alpha) > 0$.
Having established that $f_2(\alpha) > 0$ always holds in our case, where $\alpha > \beta$, we try to determine the sign of the factor $f_3(\alpha)$ in (4.4) and (4.5). Since $f_2(\alpha) > 0$, we have that $f_3(\alpha) \sim f_2^2(\alpha) - [4\beta(\alpha^2 + 1 - \beta^2)(\alpha^2 + \beta^2)\sqrt{\alpha^2 + \alpha^4 - \beta^4}]^2$ which, after a number of calculations using Maple 9, gives that $f_3(\alpha) \sim (\alpha^2 - \beta^2)^4(\alpha^2 + 1 + \beta^2)^4 > 0$. Consequently, taking into consideration (4.5), we have that $f(\omega_2) \sim f_1(\alpha)$. In other words,

$$f(\omega_2) \begin{cases} \leq 0 \text{ for } \alpha \in (\beta, \alpha_1], \\ \geq 0 \text{ for } \alpha \in [\alpha_1, +\infty). \end{cases} \tag{4.11}$$

Therefore, the relative position of ω_2 and $\widehat{\omega}_1$ is as follows:

$$\mathcal{C}: \ \omega_2 \leq \widehat{\omega}_1 \ \text{ for } \ (\beta <)\, \alpha \leq \alpha_1 \quad \text{and} \quad \omega_2 \geq \widehat{\omega}_1 \ \text{ for } \ \mathcal{D}: \ \alpha \geq \alpha_1\, (> \beta). \tag{4.12}$$

To order ω_2 and $\widehat{\omega}_2$, we note first that for $\alpha > 0$, $\widehat{\omega}_2 < \frac{2}{1+\alpha}$, since it is equivalent to $\frac{1}{1+\alpha^2} < \frac{2}{1+\alpha}$ or to $2\alpha^2 - \alpha + 1 > 0$, which is always true. Next, we form the difference $\widehat{\omega}_2 - \omega_2$ to successively obtain

$$\begin{aligned} \widehat{\omega}_2 - \omega_2 &= \tfrac{1}{1+\alpha^2} - \tfrac{2\beta}{\beta + \sqrt{\alpha^4 + \alpha^2 - \beta^4}} &&\sim \sqrt{\alpha^4 + \alpha^2 - \beta^4} - \beta(1 + 2\alpha^2) \\ &\sim \alpha^4 + \alpha^2 - \beta^4 - \beta^2(1 + 2\alpha^2)^2 &&= -[\beta^4 + (1 + 2\alpha^2)^2\beta^2 - \alpha^2(1 + \alpha^2)]. \end{aligned}$$

The right hand side above, as a quartic in β, has four zeros two of which are complex numbers. Of the two real zeros, which are opposite in sign, the positive one is

$$\sqrt{\frac{-(1 + 2\alpha^2)^2 + \sqrt{(1 + 2\alpha^2)^4 + 4\alpha^2(1 + \alpha^2)}}{2}}.$$

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This is proved that it strictly increases in $(0, \frac{1}{2})$ with α increasing in $(0, +\infty)$ and is also strictly less than α. Therefore we have that

$$
\begin{aligned}
&\mathcal{E}: \ \widehat{\omega}_2 \geq \omega_2 \ \text{ for } \ 0 < \beta \leq \sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \\
&\mathcal{F}: \ \widehat{\omega}_2 \leq \omega_2 \ \text{ for } \ \sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}} \leq \beta < \min\{\alpha, 1\}.
\end{aligned}
\tag{4.13}
$$

5. The optimal value $\widehat{\omega}$. In all the cases but **Subcase IIb**, examined in the previous section, we concluded that $\widehat{\omega} = \widehat{\omega}_1$. To determine $\widehat{\omega}$ in **Subcase IIb**, we have to combine the results in the last line of the summary in (3.13) together with the orderings of $\widehat{\omega}_1, \widehat{\omega}_2, \omega_2$ exhibited in (4.3), (4.12) and (4.13). For this we have to form triads taking one ordering from each pair of the aforementioned relations. Thus, we can form Table 5.1 below, where for each triad the following elements are illustrated: the ordering among $\widehat{\omega}_1, \widehat{\omega}_2, \omega_2$, the restrictions on β and α as well as on functions of them, and the optimal value $\widehat{\omega}$. The reader should have in mind two issues: i) Equalities all the way among $\widehat{\omega}_1, \widehat{\omega}_2$ and ω_2 do not hold except in a trivial case which, although it has not been examined, is included for completeness. ii) There are triads which do not lead to an acceptable case except in the sense explained in (i) previously.

Also, to explain things better we should recall that: i) At $\omega = \omega_2$ we always have $T(\beta^2, \omega)_{|\omega=\omega_2} = T(\alpha^2, \omega)_{|\omega=\omega_2}$. ii) From Theorems 3.1 and 3.2, both $\|\mathcal{L}_\omega\|_2$ and $T(\beta^2, \omega)$ strictly decrease and strictly increase in $(0, \widehat{\omega}_1]$ and in $[\widehat{\omega}_1, 1]$, respectively, while $\|\mathcal{L}_\omega\|_2$ and $T(\alpha^2, \omega)$ behave in an analogous way in $(0, \widehat{\omega}_2]$ and $[\widehat{\omega}_2, 1]$.

$(\mathcal{A}, \mathcal{C}, \mathcal{E})$: First, from the last line of (3.13) we have that $\omega_2 < \widehat{\omega}_1$. Hence $\|\mathcal{L}_\omega\|_2$ behaves in the same strictly decreasing manner in $(0, \omega_2]$ like $T(\beta^2, \omega)$. Then, because $T(\beta^2, \omega) \leq T(\alpha^2, \omega)$ in $[\omega_2, \frac{2}{1+\alpha})$ and $\omega_2 < \widehat{\omega}_2$, $\|\mathcal{L}_\omega\|_2$ goes on decreasing until $\omega = \widehat{\omega}_2$, and afterwards strictly increases. Therefore $\widehat{\omega} = \widehat{\omega}_2$. However, for this conclusion to be acceptable, we must recall the restrictions imposed on β and α. Specifically, since $\alpha_1 > \alpha_0$, from (4.3) and (4.12) we have that $\beta < \alpha < \alpha_0 < \alpha_1$. For $\beta < \alpha_0$ to hold, $\beta \in (0.81942756935, 1)$. Considering these restrictions together with the one in (4.13), we conclude that

$$
\beta \in \left(0.81942756935, \sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}}\right).
$$

However, since the expression for the upper bound for the interval of β is always in $(0, \frac{1}{2})$, as was proven in the end of the previous section, the conclusion is that such a case can not exist.

$(\mathcal{A}, \mathcal{C}, \mathcal{F})$: In this case all possible candidates for $\widehat{\omega}$ are equal meaning, among others from (4.12), (4.7) and (4.3), (4.10), that $\alpha_1 = \alpha_0$. However, since $\alpha_1 > \alpha_0$, $\forall \beta \in (0, 1)$, the only case equality can hold is when $\beta = 0$. This leads to $\alpha_0 = 0$ and since $\alpha \leq \alpha_0$, then $\alpha = 0$. This is a case **not** considered in this work. However, if we include it for completeness we can see that there is no contradiction. This is because, for $\beta \to 0^+$, $\widehat{\omega}_1 \to 1^-$, for $(\beta <)\ \alpha \to 0^+$, $\widehat{\omega}_2 \to 1^-$ and for $\beta \to \alpha \neq 0$, $\omega_2 \to 1$, as was noted at the end of section 3. Also, $T(\beta^2, \omega)_{|\beta=0} = T(\alpha^2, \omega)_{|\alpha=0} = 2(1-\omega)^2$, and the minimum of $\|\mathcal{L}_\omega\|_2$ takes place for $\widehat{\omega} = 1$, in which case $\|\mathcal{L}_{\widehat{\omega}=1}\|_2 = 0$.

$(\mathcal{A}, \mathcal{D}, \mathcal{E})$: Again, we note that $\alpha \leq \alpha_0$ and $\alpha \geq \alpha_1$, a case which holds only if we accept that $\alpha_0 = \alpha_1$ leading to $\alpha = 0$ and then to $\beta = 0$. This is consistent with the first of (4.13) provided we include $\beta = 0$ in it. So, we are led to the same situation as above.

$(\mathcal{A}, \mathcal{D}, \mathcal{F})$: We are in the same situation as above.

$(\mathcal{B}, \mathcal{C}, \mathcal{E})$: This case is analogous to that in case $(\mathcal{A}, \mathcal{C}, \mathcal{E})$. The complete proof is there-

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fore omitted. We simply note that the new restrictions on β and α are

$$0 < \beta \le \sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}}$$

and $\alpha_0 \le \alpha \le \alpha_1$. These restrictions are independent from each other in the sense that there always exist pairs (β, α) satisfying both. Regarding the optimal value $\widehat{\omega}$ note the following. First, $\|\mathcal{L}_\omega\|_2$ decreases with $T(\beta^2, \omega)$ decreasing in $(0, \omega_2]$, next goes on decreasing as $T(\alpha^2, \omega)$ decreases in $[\omega_2, \widehat{\omega}_2]$, since $\omega_2 \le \widehat{\omega}_2$, and then strictly increases. Therefore, $\widehat{\omega} = \widehat{\omega}_2$.

$(\mathcal{B}, \mathcal{C}, \mathcal{F})$: The restrictions are analogous to the ones in the previous case except that the interval for β is different. For $\widehat{\omega}$ observe, from the last line in (3.13), that $\|\mathcal{L}_\omega\|_2$ decreases with $T(\beta^2, \omega)$ decreasing in $(0, \omega_2]$, since $\omega_2 \le \widehat{\omega}_1$. Then, it strictly increases with $T(\alpha^2, \omega)$ increasing, because $\widehat{\omega}_2 \le \omega_2$. Consequently, $\widehat{\omega} = \omega_2$.

$(\mathcal{B}, \mathcal{D}, \mathcal{E})$: We are in a case where all possible optima are the same and, as in a previous similar situation, we have $\beta = \alpha = 0$ and $\widehat{\omega} = 1$

$(\mathcal{B}, \mathcal{D}, \mathcal{F})$: Since $\|\mathcal{L}_\omega\|_2$ has the same monotonic behavior as $T(\beta^2, \omega)$ in $(0, \omega_2]$,, and the latter function of ω strictly decreases in $(0, \widehat{\omega}_1]$, and then strictly increases in $[\widehat{\omega}_1, \omega_2]$, it is concluded that $\widehat{\omega} = \widehat{\omega}_1$. There is no other possible optimal since $\widehat{\omega}_2 \le \widehat{\omega}_1$ and $\|\mathcal{L}_\omega\|_2$ strictly increases with $T(\alpha^2, \omega)$. The restrictions in this case are

$$\sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}} \le \beta < \min\{\alpha, 1\} \quad \text{and} \quad \alpha \ge \alpha_1 .$$

TABLE 5.1
Optimal value $\widehat{\omega}$ for $\beta \in (0, \min\{\alpha, 1\})$.

Triad	Ordering of possible optimal ω's	Restrictions on β and α	$\widehat{\omega}$
$(\mathcal{A}, \mathcal{C}, \mathcal{E})$	$\omega_2 \le \widehat{\omega}_1 \le \widehat{\omega}_2$	Such a case can not exist	–
$(\mathcal{A}, \mathcal{C}, \mathcal{F})$	$\widehat{\omega}_2 \le \omega_2 \le \widehat{\omega}_1 \le \widehat{\omega}_2$	$\beta = \alpha = 0$	1
$(\mathcal{A}, \mathcal{D}, \mathcal{E})$	$\widehat{\omega}_1 \le \omega_2 \le \widehat{\omega}_2$	$\beta = \alpha = 0$	1
$(\mathcal{A}, \mathcal{D}, \mathcal{F})$	$\widehat{\omega}_1 \le \widehat{\omega}_2 \le \omega_2$	$\beta = \alpha = 0$	1
$(\mathcal{B}, \mathcal{C}, \mathcal{E})$	$\omega_2 \le \widehat{\omega}_2 \le \widehat{\omega}_1$	$0 < \beta < \sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}}$ $\alpha_0 \le \alpha \le \alpha_1$	$\widehat{\omega}_2$
$(\mathcal{B}, \mathcal{C}, \mathcal{F})$	$\widehat{\omega}_2 \le \omega_2 \le \widehat{\omega}_1$	$\sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}} < \beta < \min\{\alpha, 1\}$ $\alpha_0 \le \alpha \le \alpha_1$	ω_2
$(\mathcal{B}, \mathcal{D}, \mathcal{E})$	$\widehat{\omega}_1 \le \omega_2 \le \widehat{\omega}_2 \le \widehat{\omega}_1$	$\beta = \alpha = 0$	1
$(\mathcal{B}, \mathcal{D}, \mathcal{F})$	$\widehat{\omega}_2 \le \widehat{\omega}_1 \le \omega_2$	$\sqrt{\frac{-(1+2\alpha^2)^2 + \sqrt{(1+2\alpha^2)^4 + 4\alpha^2(1+\alpha^2)}}{2}} < \beta < \min\{\alpha, 1\}$ $\alpha \ge \alpha_1$	$\widehat{\omega}_1$

The summary of the results of this and the previous sections, where the extreme cases $\beta > \alpha = 0$ and $\alpha > \beta = 0$ are also incorporated, is given in Theorem 1.1.

6. Concluding remarks and discussion. In this concluding section we make a number of points regarding other optimal ω's that are found in the literature under the main assumptions of the present work. Usually, what many researchers minimize is either the spectral radius of the SOR operator $\mathcal{L}_\omega$, or an equivalent quantity to it, over the real parameter ω. Although $\rho(\mathcal{L}_\omega) \leq \|\mathcal{L}_\omega\|_2$ it is worth to compare $\widehat{\omega}$ of this work against other optimal values of ω.

First, it was Wrigley [10] (see also [11]) who found that the optimal value of ω, denoted by $\widehat{\omega}_W$, that minimizes the spectral radius of the SOR operator $\mathcal{L}_\omega$ over all real ω's, is

$$\widehat{\omega}_W \;=\; \frac{2}{1+\sqrt{1-\beta^2+\alpha^2}}, \quad \rho(\mathcal{L}_{\widehat{\omega}_W}) \;=\; \left(\frac{\alpha+\beta}{1+\sqrt{1-\beta^2+\alpha^2}}\right)^2. \tag{6.1}$$

Then, Eiermann, Li and Varga [1] studied a number of methods and introduced, among others, a hybrid one for the solution of the Jacobi fixed-point equation

$$x \;=\; Bx + c. \tag{6.2}$$

In the limit, the method they proposed is a stationary two-step method with coefficients functions of β and α only. For the aforementioned method they obtained an asymptotic average convergence factor given by

$$\kappa(\mathcal{C}_{\beta,\alpha}) \;=\; \frac{\sqrt{1+\alpha^2}-\sqrt{1-\beta^2}}{\sqrt{\alpha^2+\beta^2}}. \tag{6.3}$$

Finally, Eiermann and Varga [2], by applying semiiteration directly to the corresponding SOR fixed-point equation

$$x \;=\; \mathcal{L}_\omega x + \omega \begin{bmatrix} I_p & O_{p,q} \\ -\omega N & I_q \end{bmatrix}^{-1} c, \tag{6.4}$$

found as an asymptotic average convergence factor the square of the one in (6.3). Namely

$$\kappa(\Omega_{\omega,\beta,\alpha}) \;=\; \frac{(\sqrt{1+\alpha^2}-\sqrt{1-\beta^2})^2}{\alpha^2+\beta^2} \;<\; \rho(\mathcal{L}_{\widehat{\omega}_W}), \tag{6.5}$$

$$\forall \omega \in \left[\frac{2}{1+\sqrt{1+\alpha^2}}, \frac{2}{1+\sqrt{1-\beta^2}}\right] =: I_{\omega,\beta,\alpha},$$

where it is noted that $\widehat{\omega}_W \;\in\; I_{\omega,\beta,\alpha}$.

It is interesting to examine and find out whether the optimal ω, $\widehat{\omega}$, of the present work is close to $\widehat{\omega}_W$ of (6.1) or at least lies in the interval $I_{\omega,\beta,\alpha}$ above. For this the cases of Table 1.1 are examined from the simplest to the most complicated one.

i) $\widehat{\omega} = 1$: In this trivial case ($\beta = \alpha = 0$) we have

$$\widehat{\omega} \;=\; \widehat{\omega}_2 \;=\; \omega_2 \;=\; \widehat{\omega}_1 \;=\; 1 \;=\; \widehat{\omega}_W.$$

ii) $\widehat{\omega} = \widehat{\omega}_2$: In these cases it is readily checked that

$$\widehat{\omega}_2 \;=\; \frac{1}{1+\alpha^2} \;<\; \frac{2}{1+\sqrt{1+\alpha^2}} \;\leq\; \frac{2}{1+\sqrt{1-\beta^2+\alpha^2}} \;=\; \widehat{\omega}_W. \tag{6.6}$$

In other words, for a fixed β in its restriction interval(s), the bigger α is in its corresponding interval, the farther away to the left from the left endpoint of $I_{\omega,\beta,\alpha}$, and therefore from $\widehat{\omega}_W$, $\widehat{\omega}_2$ is.

iii) $\widehat{\omega} = \omega_2$: It can directly be checked that

$$\omega_2 = \frac{2\beta}{\beta + \sqrt{\alpha^4 + \alpha^2 - \beta^4}} < \frac{2}{1 + \sqrt{1 - \beta^2 + \alpha^2}} = \widehat{\omega}_W, \tag{6.7}$$

since this is equivalent to $\beta < \alpha$ which always holds in the case under examination. To find the relative position of ω_2 with respect to the left endpoint of the interval $I_{\omega,\beta,\alpha}$, $\frac{2}{1+\sqrt{1+a^2}}$, we form their difference and try to find when it is nonnegative

$$\omega_2 - \frac{2}{1 + \sqrt{1 + a^2}} \sim \beta^4 + (1 + \alpha^2)\beta^2 - \alpha^2(1 + \alpha^2) \geq 0 \tag{6.8}$$

or, equivalently,

$$\beta \geq \sqrt{\frac{-(1 + \alpha^2) + \sqrt{(1 + \alpha^2)^2 + 4\alpha^2(1 + \alpha^2)}}{2}} =: \gamma(\alpha). \tag{6.9}$$

Comparing $\gamma(\alpha)$ with the lower bound for β we find successively that

$$\begin{aligned}
&\sqrt{\tfrac{-(1+\alpha^2)+\sqrt{(1+\alpha^2)^2+4\alpha^2(1+\alpha^2)}}{2}} - \sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}} \\
&\sim [(1 + 2\alpha^2)^2 - (1 + \alpha^2)] - [\sqrt{(1 + 2\alpha^2)^4 + 4\alpha^2(1 + \alpha^2)} - \sqrt{(1 + \alpha^2)^2 + 4\alpha^2(1 + \alpha^2)}] \\
&\sim \sqrt{[(1 + 2\alpha^2)^4 + 4\alpha^2(1 + \alpha^2)][(1 + \alpha^2)^2 + 4\alpha^2(1 + \alpha^2)]} - (1 + \alpha^2)[(1 + 2\alpha^2)^2 + 4\alpha^2] \\
&\sim [(1 + 2\alpha^2)^4 + 4\alpha^2(1 + \alpha^2)][(1 + \alpha^2)^2 + 4\alpha^2(1 + \alpha^2)] - (1 + \alpha^2)^2[(1 + 2\alpha^2)^2 + 4\alpha^2]^2 \\
&= 4\alpha^6(1 + \alpha^2)(3 + 4\alpha^2)^2 > 0.
\end{aligned}$$

Since $\gamma(\alpha)$ in (6.9) is strictly greater than the left bound for β we have to see whether it is strictly less than its upper bound $\min\{1, \alpha\}$. It can directly be checked that $\gamma(\alpha) < \alpha$ always holds while $\gamma(\alpha) < 1$ holds if and only if $\alpha < \sqrt[4]{2}$. Recall, from section 4, that both α_0 and α_1 are strictly increasing functions of $\beta \in [0, 1)$. From (4.2) it is also found that $\alpha_0 = 1$ if $\beta = 0.87546301112766$, while from (4.7) it is $\alpha_1 = 1$ if $\beta = 0.73479514513748$. In addition, it can be found that $\alpha_0 = \sqrt[4]{2}$ if $\beta = 0.91606894763539$, while $\alpha_1 = \sqrt[4]{2}$ if $\beta = 0.79867023837696$. Having all this in mind, we end up with the following general conclusions:

$$\begin{aligned}
&\beta \in \left[\sqrt{\tfrac{-(1+\alpha^2)+\sqrt{(1+\alpha^2)^2+4\alpha^2(1+\alpha^2)}}{2}},\ \min\{\alpha,\, 1\}\right) \\
&\wedge\ \alpha \in [\alpha_0,\, \alpha_1]\ \wedge\ \alpha < \sqrt[4]{2} \implies \omega_2 \in I_{\omega,\beta,\alpha}
\end{aligned} \tag{6.10}$$

and
(6.11)

$$\begin{aligned}
&\beta \in \left[\sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \min\left\{1,\ \sqrt{\tfrac{-(1+\alpha^2)+\sqrt{(1+\alpha^2)^2+4\alpha^2(1+\alpha^2)}}{2}}\right\}\right) \\
&\qquad \wedge\ \alpha \in [\alpha_0,\, \alpha_1] \implies \omega_2 \notin I_{\omega,\beta,\alpha}.
\end{aligned}$$

In the second case above, note that for an admissible α, the closer to the left endpoint of the interval of definition β is the farther away to the left from the left endpoint of $I_{\omega,\beta,\alpha}$, ω_2 is.

iv) $\widehat{\omega} = \widehat{\omega}_1$: In this case we do not have an explicit expression for $\widehat{\omega}_1$, as we had in the previous three cases for $\widehat{\omega}_2$ and ω_2. So, we base our analysis directly on the restriction intervals of Table 1.1. Three subcases are to be distinguished that are examined from the simplest to the most complicated one.

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iva) $0 < \beta = \alpha < 1$: In this case $\widehat{\omega}_1 < 1 = \widehat{\omega}_W$. To compare $\widehat{\omega}_1$ against the left endpoint of the interval of $I_{\omega,\beta,\alpha}$, which can be written as $\frac{2}{1+\sqrt{1+\beta^2}}$, it suffices to find the sign of $(1+\sqrt{1+\beta^2})^4 f(\frac{2}{1+\sqrt{1+\beta^2}})$, where f is the function in (3.1). Using Maple we can find that

$$(1+\sqrt{1+\beta^2})^4 f\left(\frac{2}{1+\sqrt{1+\beta^2}}\right) \sim 9\beta^4+3\beta^2-1, \tag{6.12}$$

from which it is obtained that

$$\beta=\alpha \in \left(0, \sqrt{\frac{-1+\sqrt{5}}{6}}\right] \implies \widehat{\omega}_1 \in I_{\omega,\beta,\alpha}, \tag{6.13}$$

$$\beta=\alpha \in \left(\sqrt{\frac{-1+\sqrt{5}}{6}}, 1\right) \implies \widehat{\omega}_1 \notin I_{\omega,\beta,\alpha}. \tag{6.14}$$

In the second case above, the closer $\beta\ (=\alpha)$ gets to 1, the farther away $\widehat{\omega}_1$ gets from the left endpoint of $I_{\omega,\beta,\alpha}$.

ivb) $0 \le \alpha < \beta < 1$: To find the relative position of $\widehat{\omega}_1$ with respect to $\widehat{\omega}_W$ for any admissible pair (β,α), we form the expression below and, using Maple, we obtain

$$\begin{aligned}(1+\sqrt{1-\beta^2+\alpha^2})^4 f\left(\frac{2}{1+\sqrt{1-\beta^2+\alpha^2}}\right) &\sim \\ (\beta^2-1)\alpha^4+(2\beta^4+2\beta^2-1)\alpha^2+(\beta^6+3\beta^4+\beta^2).\end{aligned} \tag{6.15}$$

Equating the rightmost expression of (6.15) with zero and solving for α^2, we find the two roots $\frac{-1+2\beta^2+2\beta^4\pm\sqrt{1+8\beta^4}}{2(1-\beta^2)}$, one of which is negative and the other positive for all $\beta \in (0,1)$. From the positive root we can find that we always have $\beta < \sqrt{\frac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}$. Therefore, for all admissible values of α it is implied that $\widehat{\omega}_1 < \widehat{\omega}_W$.
The left endpoint of $I_{\omega,\beta,\alpha}$ is $\frac{2}{1+\sqrt{1+\alpha^2}}$, and so we have to compare it with $\widehat{\omega}_1$. We find $(1+\sqrt{1+\alpha^2})^4 f\left(\frac{2}{1+\sqrt{1+\alpha^2}}\right)$, since we are mainly interested when $\widehat{\omega}_1 \in I_{\omega,\beta,\alpha}$. Using Maple again we end up with

$$(1+\sqrt{1+\alpha^2})^4 f\left(\frac{2}{1+\sqrt{1+\alpha^2}}\right) \sim (\beta^2-1)\alpha^4+(4\beta^4-1)\alpha^2+4\beta^4(1+\beta^2). \tag{6.16}$$

Working as above and solving for α^2 we obtain the two roots $\frac{-1+4\beta^4\pm\sqrt{1+8\beta^4}}{2(1-\beta^2)}$, in terms of β, one of which is negative and the other positive for $\beta \in (0,1)$. From the latter, two opposite in sign real roots for α are obtained. For the relative position of the positive root and β we can find that $\sqrt{\frac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} \le \beta$ if and only if $\beta \in \left(0, \sqrt{\frac{-1+\sqrt{5}}{6}}\right]$. Combining this result with the sign of the rightmost expression in (6.16) when β takes values within the two real roots or outside their interval we have the following conclusions,

$$\alpha \in \left[\sqrt{\frac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}, \beta\right) \wedge \beta \in \left(0, \sqrt{\frac{-1+\sqrt{5}}{6}}\right) \implies \widehat{\omega}_1 \in I_{\omega,\beta,\alpha}, \tag{6.17}$$

while

$$\alpha \in \left[0,\ \min\left\{\beta,\ \sqrt{\frac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}\right\}\right) \wedge \beta \in (0,1) \implies \widehat{\omega}_1 \notin I_{\omega,\beta,\alpha}. \tag{6.18}$$

Observe that in the last case above (6.18), for a fixed β, the closer to 0, α is the farther away from the left endpoint of $I_{\omega,\beta,\alpha}$, $\widehat{\omega}_1$ is.

ivc) $\beta \in \left(\sqrt{\frac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \min\{\alpha,1\}\right) \wedge\ \alpha > \alpha_1$: Working as in the previous subcase (ivb) by considering the expression in the left hand side of (6.15), we end up with quite different conclusions regarding the relative position of $\widehat{\omega}_1$ with respect to $\widehat{\omega}_W$. This will be examined after the examination of the position of $\widehat{\omega}_1$ with respect to $I_{\omega,\beta,\alpha}$ takes place.

So, considering the expression $(1+\sqrt{1+\alpha^2})^4\, f\left(\frac{2}{1+\sqrt{1+\alpha^2}}\right)$, as in the previous subcase, we conclude that whenever $\sqrt{\frac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} < \alpha$ it is implied that $\widehat{\omega}_1 \in I_{\omega,\beta,\alpha}$. To go on, we have to compare the two lower bounds for α. It can be obtained that, for all $\beta \in (0,1)$,

$$\begin{aligned}
&\sqrt{\tfrac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} - \alpha_1 \ \sim\ \tfrac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)} - \tfrac{1}{4}\left(-2+2\sqrt{\tfrac{1+\beta^2+4\beta^6+2\sqrt{\beta^4+8\beta^8}}{1-\beta^2}}\right) \sim \\
&(-\beta^2+4\beta^4+\sqrt{1+8\beta^4})^2 \ -\ (1+\beta^2+4\beta^6+2\sqrt{\beta^4+8\beta^8})(1-\beta^2) \ \sim \\
&5\beta^2-6\beta^4-2\sqrt{1+8\beta^4}+10\beta^6+5\beta^2\sqrt{1+8\beta^4} \ \sim\ 25\beta^6-5\beta^4+4\beta^2-1 \ =:\ \delta(\beta).
\end{aligned}$$

It can be found that the derivative of $\delta(\beta)$ is positive, for $\beta \neq 0$. Then, applying Descartes' rule of signs, it is found that $\delta(\beta)$ has only one real zero, let it be β_1. Specifically, $\beta_1 = 0.486683413504984 \in (0,1)$. Therefore, we eventually obtain that

$$\begin{aligned}
&\beta \in \left(\sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \min\{\alpha,1\}\right) \\
&\wedge\ \alpha\ >\ \max\left\{\alpha_1,\ \sqrt{\tfrac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}\right\} \quad\implies\quad \widehat{\omega}_1 \in I_{\omega,\beta,\alpha},
\end{aligned} \tag{6.19}$$

while

$$\begin{aligned}
&\beta \in \left(\sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \min\{\alpha,1\}\right)\ \wedge\ \ \beta \in (\beta_1,1) \\
&\wedge\ \alpha \in \left(\alpha_1,\ \sqrt{\tfrac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}\right) \quad\implies\quad \widehat{\omega}_1 \notin I_{\omega,\beta,\alpha}.
\end{aligned} \tag{6.20}$$

Also, in the last case, the closer β is to its upper bound, the farther away is $\widehat{\omega}_1$ from the left endpoint of $I_{\omega,\beta,\alpha}$. To find the relative position of $\widehat{\omega}_1$ with respect to $\widehat{\omega}_W$, when the pairs (β,α) satisfy (6.19), we consider relationship (6.15) and find out whether $\widehat{\omega}_1$ can coincide with or can lie to the right of $\widehat{\omega}_W$. If such cases exist, then for all other admissible pairs (β,α), $\widehat{\omega}_1$ will be on the left of $\widehat{\omega}_W$. From (6.15) we have that for $\alpha \geq \sqrt{\frac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}$, $\widehat{\omega}_1 \geq \widehat{\omega}_W$. For the admissible lower bounds for α in (6.19) we have that

$$\sqrt{\frac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} > \sqrt{\frac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}},$$

for all $\beta \in (0,1)$. Moreover, considering the difference below and doing some simple calculations using Maple we obtain

$$\sqrt{\frac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} - \alpha_1 \sim 5+4\beta^4+3\sqrt{1+8\beta^4} > 0.$$

As a consequence from all the results of the comparisons above it is implied that for all admissible values of $\beta \in (0,1)$ for which $\widehat{\omega}_1 \in I_{\omega,\beta,\alpha}$ there hold

(6.21)

$$\beta \in \left(\sqrt{\tfrac{-(1+2\alpha^2)^2+\sqrt{(1+2\alpha^2)^4+4\alpha^2(1+\alpha^2)}}{2}}, \min\{\alpha, 1\}\right)$$

$$\wedge \begin{cases} \alpha \in \left(\max\left\{\alpha_1, \sqrt{\tfrac{-1+4\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}\right\}, \sqrt{\tfrac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}\right) & \Longrightarrow \quad \widehat{\omega}_1 < \widehat{\omega}_W, \\ a = \sqrt{\tfrac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}} & \Longrightarrow \quad \widehat{\omega}_1 = \widehat{\omega}_W, \\ a \in \left(\sqrt{\tfrac{-1+2\beta^2+2\beta^4+\sqrt{1+8\beta^4}}{2(1-\beta^2)}}, +\infty\right) & \Longrightarrow \quad \widehat{\omega}_1 > \widehat{\omega}_W. \end{cases}$$

Note that, except for the trivial case (i) where $\widehat{\omega} = 1 = \widehat{\omega}_W$, the last two subcases in (6.21) are the only ones where the optimal value of the relaxation parameter $\widehat{\omega} \in I_{\omega,\beta,\alpha}$ can equal to or, more interestingly, can lie strictly to the right of $\widehat{\omega}_W$.

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