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ON THE EXACT ESTIMATES OF THE BEST SPLINE APPROXIMATIONS OF FUNCTIONS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. In the paper the exact (in the sense of the order of smallness) estimates of the best spline approximations of functions of one variable from different functional classes on a finite segment in uniform and integral metrics are obtained.

Key words. spline, polynomial spline, best spline approximation, uniform and integral metrics, class of convex function, class of function with convex derivatives, class of function with generalized finite variation, module of continuity, module of variation, spline of the minimal defect with free knots

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1. Introduction. For the functions with singularities or with non-great smoothness the splines are a more natural apparatus of approximation than polynomials and rational functions. It is also confirmed by comparison of the results of spline approximations of this paper, with well-known polynomial and rational approximations of functions, from considering functional classes. The paper is devoted to the exact (in the sense of the order of smallness) estimates of the best approximations by polynomial splines of the minimal defect with free knots on a finite segment in uniform and integral metrics of functions from the following functional classes:

a) the class of all convex functions, satisfying the Lipschitz-Hölder condition;
b) the class of all functions with convex derivatives;
c) the class of all functions with generalized finite variation.

In order to expound the results of the paper we need in some definitions and notations.

2. Definitions and notations. Let $\mathbf{N}$ be the set of all natural numbers, $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$, $f(x)$ be a measurable with respect to Lebesgues measure on a finite segment $\Delta = [a, b]$ real-valued function, $|\Delta| = b - a$, $L_p(\Delta)$ be the space of all measurable with respect to Lebesgues measure real-valued functions on Δ, integrable with pth-power by Lebesgue and possessing with the quasi-norm

$$\|f\|_{p,G} := \left(\int_\Delta |f(x)|^p \, dx \right)^{1/p} (0 < p < \infty), \|f\|_{\infty,\Delta} = ess \sup \{|f(x)| : x \in \Delta\}.$$

For continuous function f on Δ and any real number $\delta \geq 0$ we define the modulus of continuity of function f as follows:

$$\omega(\delta, f) := \sup \{|f(x') - f(x'')| : x', x'' \in \Delta, |x' - x''| \leq \delta\}.$$

If $\omega(\delta, f) \leq K\delta^\alpha$ for all $\delta \geq 0$ and some $\alpha \in (0, 1]$, $K = \text{const} > 0$, then we say that f satisfies a Lipschitz-Hölder condition of the order α with constant K and write $f \in Lip_K \alpha$.

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Let a function $\Phi(u)$ be continuous, increasing, convex to down on $(0,\infty], \Phi(0)=0$. For a function $f(x)$ defined and finite on a segment Δ the value

$$V_\Phi(f)=V_\Phi(f,\Delta)=\sup\left\{\sum_{k=0}^{n-1}\Phi\left(\left|f\left(x_{k+1}\right)-f\left(x_k\right)\right|\right)\right\},$$

where the upper bound is taken over all possible partitions $a=x_0<x_1<\cdots<x_n=b\ (n=1,2,\ldots)$ of the segment Δ, is called Φ-variation of the function f on the segment Δ [1].

For $M>0$ we set $V_\Phi(M,\Delta)=\{f: V_\Phi(f,\Delta)\le M\}$. By definition for $\Phi(u)=u$ we have the ordinary bounded variation $V(f,\Delta)$ of the function f on a segment Δ. The value

$$\chi(f,n)=\sup\left\{\sum_{k=0}^{n-1}\left|f\left(x_{k+1}\right)-f\left(x_k\right)\right|: a\le x_0\le x_1\le\cdots\le x_n\le b\right\}$$

is called the modulus of variation of the function f on a segment Δ [2], [3].

We denote by $Conv^{(l)}(M,\Delta)$ the set of all functions f, having lth-order convex derivative on Δ with the norm $\left\|f^{(l)}\right\|_{C(\Delta)}\le M=\text{const}>0$, by $Conv^{(l)}H^\alpha(K,\Delta)$ the set of all functions f, having lth-order convex derivative $f^{(l)}\in Lip_K\alpha$ for some $\alpha, 0<\alpha\le 1$, and $K=\text{const}>0$.

A function s is called a polynomial spline (or shorter spline) of degree m of minimal defect with free $N+1$ knots $a=x_0<x_1<x_2<\cdots<x_N=b$ on a segment Δ if

1) s is polynomial of the degree non-exceeding m on each segment $[x_i,x_{i+1}], i=0,1,\ldots,N-1$;

2) the $(m-1)$th-order derivative of the function s is continuous on the segment Δ.

We denote by $S(m,N,\Delta)$ the set of all splines of degree m of minimal defect with arbitrary $N+1$ knots on a segment Δ and by $S_N^m(f,\Delta)_p$ — the least deviation of a function f with finite Φ–variation from the splines $s\in S(m,N,\Delta)$ with respect to the quasi-norm of the space $L_p(\Delta)$ i.e.,

$$S_N^m(f,\Delta)_p=\inf\left\{||f-s||_{p,\Delta}:\ s\in S(m,N,\Delta)\right\}.$$

By analogy we can define the same value with respect to the uniform norm which we denote by $S_N^m(f,\Delta)$.

We set $S_N^m(X)_p=\sup\left\{S_N^m(f,\Delta)_p: f\in X\right\}$, where X is some set of functions f.

Everywhere below $C(\alpha,\beta,\ldots), C_1(\alpha,\beta,\ldots),\ldots$ denote positive variables depending from the parameters indicated in parenthesis and from the subscripts only.

3. The main results of the paper. One of the first important results in theory of spline approximation was obtained by G. Freud and V. A. Popov [4]. They showed that if $l\ge 0$ a function $f\in Conv^{(l)}H^1(K,\Delta)$, then for all $N\in\mathbf{N}$ we have estimates

$$S_N^{l+1}(f,\Delta)\le C(l)\,K\,|\Delta|^{l+1}N^{-l-2}. \tag{3.1}$$

The main results of the present paper are the following theorems.

THEOREM 3.1. *Let Δ be a finite segment, $K=\text{const}>0, 0<\alpha\le 1, 0<p\le\infty$. Then as $N\to\infty$ the relation*

$$S_N^1\left(ConvH^\alpha(K,\Delta)\right)_p\succ\prec N^{-2}$$

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holds.

THEOREM 3.2. *If Δ is a finite segment, $M = \text{const} > 0, l \in \mathbf{Z}_+, 0 < p \leq \infty$ (at $l = 0$ $p \neq \infty$), then as $N \to \infty$ we have*

$$S_N^{l+1}\left(Conv^{(l)}\left(M, \Delta\right)\right)_p \succ\prec N^{-l-2}.$$

The lower estimates of this relation are valid also at $l = 0,\ p = \infty$.

THEOREM 3.3. *For any measurable to respect of Lebesgue's measure and bounded on a segment Δ function f and any $p,\ 0 < p < \infty,\ N \in \boldsymbol{N}$ the estimates*

$$S_N^1\left(f, \Delta\right)_p \leq 5\left|\Delta\right|^{1/p} \chi\left(f, N\right)/N$$

hold.

On the other hand, for each $N \in \boldsymbol{N}$ there is a measurable (with respect to Lebesgue measure) function $f = f_N$ bounded on a segment Δ such that for any p, $0 < p \leq \infty$,

$$S_N^1\left(f, \Delta\right)_p \geq \left(12e6^{1/p+1}\right)^{-1} \chi\left(f, N\right)/N.$$

THEOREM 3.4. *For all positive numbers p and $N \in \boldsymbol{N}$ the estimates*

$$\left(12e6^{1/p}\right)^{-1}\left|\Delta\right|^{1/p} \Phi^{-1}\left(M/N\right) \leq S_N^1\left(V_\Phi\left(M, \Delta\right)\right)_p \leq 5\left|\Delta\right|^{1/p} \Phi^{-1}\left(M/N\right)$$

are valid. The lower estimates hold for $p = \infty$ too.

REMARK 3.5. *Theorems 3.1 and 3.2 are proved in the articles [5], [6].*

The proofs of the upper estimates of Theorems 3.1 and 3.2 are based on an application of the corresponding iterative lemmas, allowing to improve a rate of convergence to zero of the best spline approximations step by step. This method can be generalized to rational approximations of functions in one and many variables with convex derivatives too (see [7], [8]). Also we emphasize that we can simplify many of the problems. For example, for the proof of the upper estimates of Theorem 3.1 at $p = \infty$, it is enough to prove it for $p = \infty$ for functions $\varphi \in Lip_{K(\varphi)}\alpha$, giving on the segment $[0,\ 1]$, convex to up, non-decreasing, continuously differentiable and such that $\varphi\left(0\right) = 0, \varphi\left(1\right) = 1$. So denoting the set of all such functions by $ConvH^\alpha\left(K\right)$ we note that for any function $\varphi \in ConvH^\alpha\left(K\right)$ and $N \in \mathbf{N}$ estimates

$$S_N^1\left(\varphi, [0, 1]\right) \leq 1 \leq K\left(\varphi\right) N^{-2+2(1-\alpha)^0} \tag{3.2}$$

hold, since $1 = \varphi\left(1\right) - \varphi\left(0\right) \leq K\left(\varphi\right)\left(1 - 0\right)^\alpha = K\left(\varphi\right)$.

In this case this is accomplished with the following iterative lemma.

LEMMA 3.6. *Suppose that there exists a value $C\left(\alpha\right) > 1$ such that for any function $\varphi \in ConvH^\alpha\left(K\right)$, any $N \in \boldsymbol{N}$ and some $k \in \mathbf{Z}_+$ the inequality*

$$S_N^1\left(\varphi, [0, 1]\right) \leq C\left(\alpha\right) K\left(\varphi\right) N^{-2+2(1-\alpha)^k}$$

holds. Then there is a constant $C \geq 1$ such that for any $N \in \boldsymbol{N}$ inequality

$$S_N^1\left(\varphi, [0, 1]\right) \leq C(C\left(\alpha\right))^{1-\alpha} K\left(\varphi\right) N^{-2+2(1-\alpha)^{k+1}}$$

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Proof. Let

$$\mu = 1/C(\alpha)N^{2(1-\alpha)^k} \tag{3.3}$$

and φ be any function from the set $ConvH^\alpha(K)$. We consider the function

$$\psi(y) = \varphi(\mu y)/\varphi(\mu), y \in [0,1]. \tag{3.4}$$

Obviously that $\psi \in ConvH^\alpha(\mu^\alpha K(\varphi)/\varphi(\mu))$. Therefore by condition of Lemma 3.6 we have inequality at $N = 1, 2, \ldots$

$$S_N^1(\psi, [0,1]) \leq C(\alpha)\mu^\alpha(K(\varphi)/\varphi(\mu))N^{-2+2(1-\alpha)^k}. \tag{3.5}$$

According to (3.3), (3.4) the inequality (3.5) we write in view

$$S_N^1(\varphi, [0,\mu]) \leq (C(\alpha))^{1-\alpha}K(\varphi)N^{-2+2(1-\alpha)^{k+1}}.$$

Consequently, there exists a function $s \in S(1, N, [0,\mu])$ such that $s(\mu) = \varphi(\mu)$and

$$\|\varphi - s\|_{C[0,\mu]} \leq 2(C(\alpha))^{1-\alpha}K(\varphi)N^{-2+2(1-\alpha)^{k+1}}. \tag{3.6}$$

Obviously that $\varphi \in ConvH^1(K_1(\varphi), [\mu, 1])$ with $K_1(\varphi) \leq \varphi(\mu)/\mu \leq K(\varphi)\mu^{\alpha-1}$. So using inequality (3.1) of G. Freud and V. A. Popov we obtain existence for each $N \in \mathbf{N}$ of a function $s_1 \in S(1, N, [\mu, 1])$ such that $s_1(\mu) = \varphi(\mu)$ and

$$\|\varphi - s_1\|_{C[\mu,\,1]} \leq C_1 K(\varphi)\mu^{\alpha-1}N^{-2}, C_1 = \text{const} \geq 1. \tag{3.7}$$

Substituting the expression (3.3) of μ into (3.7) for any $N \in \mathbf{N}$ we obtain the inequality

$$\|\varphi - s_1\|_{C[\mu,1]} \leq C_1(C(\alpha))^{1-\alpha}K(\varphi)N^{-2+2(1-\alpha)^{k+1}}. \tag{3.8}$$

From inequalities (3.6) and (3.8) it follows that for function

$$s_2(x) = \begin{cases} s(x) & for \ \ 0 \leq x \leq \mu \\ s_1(x) & for \ \ \mu \leq x \leq 1 \end{cases} \ \in S(1, 2N, [0,1]) \ \ (N = 1, 2, \ldots)$$

the estimate

$$\|\varphi - s_2\|_{C[0,1]} \leq 2C_1(C(\alpha))^{1-\alpha}K(\varphi)N^{-2+2(1-\alpha)^{k+1}} \tag{3.9}$$

is valid. In inequality (3.9) we take $N = [m/3] + 1$, then for all $m \geq 6$ we have

$$2N \leq 2(m/3 + 1) \leq m$$

and so the inequality

$$S_m^1(\varphi, [0,1]) \leq 18C_1(C(\alpha))^{1-\alpha}K(\varphi)m^{-2+2(1-\alpha)^{k+1}} \tag{3.10}$$

holds. Since $K(\varphi) \geq 1$ and $C(\alpha) > 1, C_1 \geq 1$ (see (3.2), Lemma 3.6 and (3.7)), then for $1 \leq m \leq 5$ we have the estimate

$$S_m^1(\varphi, [0,1]) \leq 12{,}5C_1(C(\alpha))^{1-\alpha}K(\varphi)m^{-2+2(1-\alpha)^{k+1}}. \tag{3.11}$$

Inequalities (3.10) and (3.11) show that Lemma 3.6 is proved. □

Obviously by consecutively using Lemma 3.6 (at first to (3.2), then to subsequent inequalities step by step) we easily obtain the upper estimates of Theorem 3.1 for the considered case.

The upper estimates of Theorem 3.2 for $l = 0$ can be proved analogously by using the following iterative lemma.

LEMMA 3.7. *Let φ be an arbitrary convex upwards, non-decreasing, continuously differentiable, function on $[0,1]$ and such that $\varphi(0) = 0$, $\varphi(1) = 1$. Suppose that $S(\varphi, N, [0,1])$ is the set of all splines $s \in S(1, N, \Delta)$ interpolating of function φ in its knots. If for some $k \in \mathbf{Z}_+$ and each $N \in \boldsymbol{N}$, $p \geq 1$ there exists $s_1 \in S(\varphi, N, [0,1])$ such that*

$$\|\varphi - s_1\|_{p,[0,1]} \leq C(p)\, N^{-2+2(p/(p+1))^k},$$

where $C(p) \geq 1$, then for each $N \in \boldsymbol{N}$ there exists such function $s_2 \in S(\varphi, N, [0,1])$ that

$$\|\varphi - s_2\|_{p,[0,1]} \leq CC(p)^{p/(p+1)}\, N^{-2+2(p/(p+1))^{k+1}},$$

where $C = \text{const} \geq 1$.

Lemma 3.7 can be established similarly to Lemma 3.6.

The proof of the upper estimates of Theorem 3.2 for $l \geq 1$ are based on using the inequality

$$S_N^{l+1}(\varphi, [0,1]) \leq C_1(l) N^{-l-2}, \; N = 1, 2, \ldots$$

for functions $\varphi(x), x \in [0,1]$, having lth-order convex derivatives $\varphi^{(l)}(x), x \in [0,1]$, such that $\varphi^{(l)}(0) = 0$, $\varphi^{(l)}(1) = 1$. The last inequality can be easily established using the following lemma proved in the article of G. Freud and V. A. Popov [4].

LEMMA 3.8. *Let $\psi \in L_1([0,1])$, $m \in \boldsymbol{N}$, $s \in S(m, N, [0,1])$ and*

$$\int_0^1 |\psi(x) - s(x)|\, dx \leq \eta.$$

Then there exists a function $s_1 \in S(m+1, 3N, [0,1])$ such that

$$\max\left\{ \left| \int_0^x \psi(t)dt - s_1(x) \right| : x \in [0,1] \right\} \leq C(m)\eta/N.$$

For proof of the upper estimate of Theorem 3.3 we use the following lemma.

LEMMA 3.9. *For all p, $0 < p < \infty$, and $n \in \boldsymbol{N}$, $M = \text{const} > 0$ we have*

$$S_n^1(V(M, \Delta))_p \leq 2\, |\Delta|^{1/p}\, M/n.$$

Proof. Since

$$S_n^1(V(M, \Delta))_p = M\, |\Delta|^{1/p}\, S_n^1(V(1, [0,1]))_p,$$

for the proof of Lemma 3.9 it is enough to establish for $\Delta = [0,1], M = 1$. Besides, for Lemma 3.9 it is enough to prove for $1 \leq p < \infty$, since if it is valid at $p = 1$, then for $0 < p < 1$ using the Hölder's integral inequality we obtain

$$S_n^1(V(M, \Delta))_p \leq |\Delta|^{\frac{1}{p}-1}\, S_n^1(V(M, \Delta))_1 \leq 2\, |\Delta|^{1/p}\, M/n.$$

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Let f be any continuous function with variation $V(f,[0,1]) = 1$ on the segment $I = [0,\ 1]$. Take a partition of the I into n parts by points $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that on each segment $I_k = [x_k, x_{k+1}]$, $k = \overline{0, n-1}$ the inequality $V(f, I_k) \le 1/n$ shall be accomplished. Then, obviously that there exists a function $s \in S(1,n,I)$ satisfying the inequality

$$\|f - s\|_{C(I)} \le 1/n. \qquad \square$$

In the general case of any Lebesgue measurable function f with bounded variation $M > 0$ the inequality of Lemma 3.9 is valid too, since changing its value on the set of arbitrary small measure, we get a continuous function f_1 with variation bounded by M.

Proof of the upper estimate of Theorem 3.3. We note that as in the case of the proof of Lemma 3.9 for Theorem 3.3 it is enough to establish for continuous function f with bounded by 1 variation on the segment $I$and $1 \le p < \infty$. Let f be an arbitrary such function and $S(0,n,I)$ be the set of all piecewise constant functions with $n+1$ knots on I. Let

$$S_n^0(f,I) = \inf_{s \in S(0,n,I)} \left\{ \sup_{x\in I} \{|f(x) - s(x)|\} \right\}$$

and $s_0 \in S(0,n,I)$ be a piecewise constant function of the best approximation of function f on the segment Iin uniform metric:

$$\|f - s_0\|_{C(I)} = S_n^0(f,I).$$

If $0 < x_1 < x_2 < \cdots < x_q < 1$ are the knots of the function s_0 $(q \le n-1)$, then

$$V(s_0, I) = \sum_{i=1}^{q} |s_0(x_i + 0) - s_0(x_i - 0)| \le 2q S_n^0(f,I) < 2n S_n^0(f,I). \tag{3.12}$$

So according to inequality (3.12) from Lemma 3.9 we get the existence of function $s_1 \in S(1,n,I)$ such that

$$\|s_0 - s_1\|_{p,I} \le 4 S_n^0(f,I). \tag{3.13}$$

By definition of s_0

$$\|f - s_0\|_{p,I} \le S_n^0(f,I). \tag{3.14}$$

Using (3.13) and (3.14) we have

$$\|f - s_1\|_{p,I} \le 5 S_n^0(f,I). \tag{3.15}$$

But $S_n^0(f,I) \le \chi(f,n)/n$ (see [3], Theorem 3.1).

Therefore in view of inequality (3.15) we obtain the upper estimate of Theorem 3.3. $\square$

As in the case of Theorems 3.1–3.3 for Theorem 3.4 it is enough to prove for $\Delta = I = [0,1]$. The upper estimate of Theorem 3.4 follows from the upper estimate of Theorem 3.3 and the inequality

$$\chi(f,n) \le n\Phi^{-1}(M/n),$$

which is valid for any $f \in V_\Phi(M,I)$.

Really, for arbitrary points $0 \le x_0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$ using the properties of function Φ we obtain

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = n\Phi^{-1}\left[\Phi(\frac{1}{n}\sum_{k=0}^{n-1}|f(x_{k+1}) - f(x_k)|)\right]$$
$$\le n\Phi^{-1}\left[\frac{1}{n}\sum_{k=0}^{n-1}\Phi(|f(x_{k+1}) - f(x_k)|)\right] \le n\Phi^{-1}\left(\frac{V_\Phi(f,I)}{n}\right).$$

The proof of the lower estimates of Theorems 3.1 and 3.2 at $1 \le p \le \infty$ is based on application of Chebishev's polynomials of the first and second kinds. To obtaining such estimates for $0 < p < 1$ we prove the special estimates

$$C\left(l,p\right)|\Delta|^{l+2+\frac{1}{p}} \le \inf\left\{\left\|x^{l+2} - g\left(x\right)\right\|_{p,\Delta} : g \in \mathbf{P}_{l+1}\right\} \le C_1\left(l\right)|\Delta|^{l+2+\frac{1}{p}},$$

where $\mathbf{P}_{l+1}$ is the set of all polynomials of the degree non-exceeding $l+1$.

To obtain the lower estimates of Theorems 3.3 and 3.4, we use a simple idea ([9], [10]) concluding in the following: if an approximatible function has enough oscillation, then any spline from the class $S\left(m,N,\Delta\right)$ is not able to approximate them and lag on a significant part of the segment Δ. For example, we can take the functions

$$f\left(x\right) = A\left(N\right)\sin\alpha Nx, f\left(x\right) = A\left(N\right)\cos\alpha Nx,$$

where $A(N)$ is some monotone decreasing to zero variable and α is a sufficiently large positive constant.

We note that under the conditions of Theorems 3.1–3.4 the approximations by rational functions of the order $\le N$ give the same rate of approximations that give the Theorems 3.1–3.4 ([7], [8], [10]). Analogously, if consider an approximation by polynomials of the degree $\le N$, then for all we know the following rates $N^{-\alpha-\frac{2-\alpha}{p}}$, $N^{-l-2/p}$, $N^{-1/p}$ correspond to Theorems 3.1–3.3 ([11], [12]).

These comparisons of the results of spline approximation of the paper with corresponding polynomial and rational approximations show that the splines with $N+1$ knots give essentially better rates of approximations than polynomials of the degree $\le N$ and the same rates of approximations that give the rational functions of the order $\le N$. But since splines have more simple structure, then we conclude that they confirm their status as a more natural apparatus of approximation.

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