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ON THE SUPPORT OF THE EQUILIBRIUM MEASURE FOR ARCS OF THE UNIT CIRCLE AND FOR REAL INTERVALS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We study the support of the equilibrium measure for weights defined on arcs of the unit circle and on intervals of the compactified real line. We provide several conditions to ensure that the support of the equilibrium measure is one interval or one arc.

Key words. logarithmic potential theory, external fields, equilibrium measure, equilibrium support

AMS subject classifications. 31A15, 30C15, 78A30

1. Introduction. In recent years, equilibrium measures with external fields have found an increasing number of applications in a variety of areas ranging from diverse subjects such as orthogonal polynomials, weighted Fekete points, numerical conformal mappings, weighted polynomial approximation, rational and Pade approximation, integrable systems, random matrix theory and random permutations. We refer the reader to the references [1, 2, 4, 7, 8, 12, 13, 14, 16, 17, 19, 20, 21] and those listed therein for a comprehensive account of these numerous, vast and interesting applications.

1.1. Potential-theoretic preliminaries and definitions. With a compact set $\Sigma \subset \mathbb{C}$ and lower semi-continuous external field $q : \Sigma \to (-\infty, \infty]$, we set $w := \exp(-q)$ and call w a weight associated with q, provided the set

$$\Sigma_0 := \{z \in \Sigma : \, w(z) > 0\}$$

has positive logarithmic capacity. With an external field q (or a weight w), we associate the weighted energy of a Borel probability measure μ on Σ as

$$I_w(\mu) = \int_\Sigma \int_\Sigma \log \frac{1}{|s-t|w(s)w(t)} d\mu(s)d\mu(t).$$

The equilibrium measure in the presence of an external field q, is the unique probability measure μ_w on Σ minimizing the weighted energy among all probability measures on Σ. Thus,

$$I_w(\mu_w) = \min\{I_w(\mu) : \mu \in \mathcal{P}(\Sigma)\},$$

where $\mathcal{P}(\Sigma)$ denotes the class

$$\mathcal{P}(\Sigma) = \{\mu : \mu \text{ is a Borel probability measure on } \Sigma\}.$$

For more details on these topics we refer the reader to the seminal monograph of E. B. Saff and V. Totik [17].

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The determination of the support S_w of the equilibrium measure μ_w is a major step in obtaining the measure. As described by Deift [8, Chapter 6], information that the support consists of $N \geq 1$ disjoint closed intervals, allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated. Knowing the endpoints, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation. It is for this reason that it is important to have a priori conditions on the external field q to ensure that the support is an interval or the union of a finite number of intervals. We refer the reader to the references [3, 4, 5, 6, 9, 10, 11, 15, 17, 18] for an account of advances on the equilibrium measure and support problem for one or several intervals.

In this present paper, we study supports of equilibrium measures for a general class of weights on the compactified real line and unit circle and present several conditions on the associated external field to ensure that the support of the associated equilibrium measure is one interval or one arc.

In order to present our main results, we find it convenient to introduce some needed notation and definitions.

DEFINITION 1.1. *Let* $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ *denote the compactified real line. It is a topological space which is isomorphic to the unit circle* C. *We will think of* ∞ *as* $+\infty$, *that is, we agree that* $a < \infty$ *for any* $a \in \mathbb{R}$.

Let $U, V \in \bar{\mathbb{R}}, U \leq V$. *Then* $I := \lfloor U, V \rceil \subset \bar{\mathbb{R}}$ *denotes an interval which is open, closed, or half open, and has endpoints* U *and* V. *We define* $[V, U] := (U, V)^c$, $(V, U) := [U, V]^c$, $(V, U] := (U, V]^c$, $[V, U) := [U, V)^c$.

Let now $\alpha, \beta \in \mathbb{R}$ *be two angles,* $|\beta - \alpha| < 2\pi$. *We define* $\widehat{\lfloor \alpha, \beta \rceil}$ *to be the arc* $\lfloor e^{i\alpha}, e^{i\beta} \rceil \subset C$, *where we go from* $e^{i\alpha}$ *to* $e^{i\beta}$ *in a counterclockwise direction. If* $\beta - \alpha = 2\pi$, *let* $\widehat{\lfloor \alpha, \beta \rceil}$ *to be the full circle C. If* $\alpha - \beta = 2\pi$, *or* $\alpha = \beta$, *then let* $\widehat{\lfloor \alpha, \beta \rceil}$ *be the single point* $\exp(i\alpha)$. *Finally, if* $0 \leq \beta - \alpha \leq 2\pi$ *and* $I = \lfloor \alpha, \beta \rceil$ *then define* $\widehat{I}$ *to be* $\widehat{\lfloor \alpha, \beta \rceil}$.

We say that $W(X)$, $X \in \bar{\mathbb{R}}$ *is a weight on* $\bar{\mathbb{R}}$, *if*

$$w(x) := \frac{W(\frac{1+x}{1-x} i)}{|1-x|}, \quad |x| = 1 \tag{1.1}$$

is a weight on C.

REMARK 1. *We note that this definition of weights on the real line is more general than the one given in [17] or [18], since we do not assume the existence of* $\lim |X| W(X)$ *as* $|X| \to \infty$. *However, since* $q := -\log(w)$ *is bounded from below,* $|X| W(X)$ *must be bounded from above. In addition, studying weights on the compactified real line via weights on the unit circle* C *allows us to deduce several results on the supports of the equilibrium measure* μ_W *on the line via a general result for* μ_w *on the circle (see Theorems 2.1, 2.3 and 2.4).*

In the next subsection, we describe the relation between the weighted energy problem on $\bar{\mathbb{R}}$ and on C.

1.2. Connection between the equilibrium problem on $\bar{\mathbb{R}}$ and on C. We will make use of the Cayley transform between $\bar{\mathbb{R}}$ and C as follows.

$$\bar{\mathbb{R}} \ni X \longmapsto x := \frac{X - i}{X + i} \in C$$

defines a bijection between $\bar{\mathbb{R}}$ and C. The inverse is

$$C \ni x \longmapsto X = \frac{1+x}{1-x} i \in \bar{\mathbb{R}}.$$

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The image of $Y, T \in \bar{\mathbb{R}}$ by the Cayley transform will be denoted by y and t.

To any measure $\mu \in \mathcal{P}(\bar{\mathbb{R}})$, we assign the Borel probability measure μ_C on C with

$$d\mu_C(x) := d\mu(X).$$

This mapping is a bijection between Borel probability measures on $\bar{\mathbb{R}}$ and C.

Let the weights W and w be related by (1.1). The weighted logarithmic potential of μ and μ_C is defined by

$$U_W^{\mu}(X) := \int \log \frac{1}{|T-X|W(T)W(X)} d\mu(T),$$

$$U_w^{\mu_C}(x) := \int \log \frac{1}{|t-x|w(t)w(x)} d\mu_C(t),$$

respectively ([18]). These are well-defined integrals (even though μ may not have compact support), as well as

$$I_W(\mu) := -\int\int \log(|X-Y|W(X)W(Y))d\mu(X)d\mu(Y).$$

From

$$|X-Y| = \left|\frac{1+x}{1-x}i - \frac{1+y}{1-y}i\right| = \frac{2|x-y|}{|1-x||1-y|}$$

we have $|T-X|W(T)W(X) = 2|t-x|w(t)w(x)$. Thus,

$$U_W^{\mu}(X) = U_w^{\mu_C}(x) - \log 2. \tag{1.2}$$

Integrating this we get

$$I_W(\mu) = I_w(\mu_C) - \log 2. \tag{1.3}$$

Since

$$W = e^{-Q}, \quad w = e^{-q},$$

we have the following correspondence between q and Q:

$$q(x) = Q\Big(\frac{1+x}{1-x}i\Big) + \log|1-x|, \quad |x| = 1. \tag{1.4}$$

For convenience we will agree on the notations

$$q(\theta) := q(e^{i\theta}),\ w(\theta) := w(e^{i\theta}), \quad \theta \in \mathbb{R}.$$

Also, since

$$|1-x| = \frac{2}{|X+i|} = \frac{2}{\sqrt{1+X^2}}, \quad |x| = 1, \quad X \in \bar{\mathbb{R}},$$

we have

$$Q(X) = q\Big(\frac{X-i}{X+i}\Big) + \frac{1}{2}\log(1+X^2) - \log 2, \quad X \in \bar{\mathbb{R}}. \tag{1.5}$$

We find it more convenient to use angles instead of complex numbers on the unit circle. So let $x = e^{i\theta}$, and $y = e^{i\nu}$ for $\theta, \nu \in \mathbb{R}$.

Clearly,

$$\frac{|x-y|}{|1-x||1-y|} = \frac{|\sin\frac{\theta-\nu}{2}|}{2|\sin\theta/2||\sin\nu/2|} \quad \text{and} \quad \frac{1+x}{1-x}i = -\cot\frac{\theta}{2}. \tag{1.6}$$

Therefore, using (1.6), we readily calculate that

$$\begin{aligned} I_W(\mu) = \\ &= -\int\int \log\left(\left|\sin\frac{\theta-\nu}{2}\right| \frac{W(-\cot\theta/2)}{|\sin\theta/2|}\frac{W(-\cot\nu/2)}{|\sin\nu/2|}\right) d\mu \\ &\quad \times \left(-\cot\frac{\theta}{2}\right) d\mu\left(-\cot\frac{\nu}{2}\right) \\ &= -\int\int \log\left(\left|\sin\frac{\theta-\nu}{2}\right| w(\theta)w(\nu)\right) d\mu\left(-\cot\frac{\theta}{2}\right) d\mu\left(-\cot\frac{\nu}{2}\right) - \log 4. \end{aligned}$$

Here, we used the fact that $w(\theta) = W(-\cot\frac{\theta}{2})/(2|\sin\frac{\theta}{2}|)$ (see (1.1)). In addition, we note that from (1.4) we get

$$q(\theta) = Q\left(-\cot\frac{\theta}{2}\right) + \log|\sin\frac{\theta}{2}| + \log 2. \tag{1.7}$$

The formulae (1.1)–(1.3) allow us to conclude the following:

$\mu \in \mathcal{P}(\bar{\mathbb{R}})$ minimizes the energy integral $I_W(\mu)$ over all probability measures on $\bar{\mathbb{R}}$ if and only if its corresponding $\mu_C \in \mathcal{P}(C)$ minimizes the energy integral $I_w(\mu_C)$ over all probability measures on C. Moreover, the support S_W is going to be an interval or a complement of an interval in $\bar{\mathbb{R}}$ if and only if the corresponding support S_w is an arc on C.

We close this section by introducing some remaining conventions which we assume henceforth.

Let $\tilde{I}$ be an arc of C. We shall say that $f : \tilde{I} \to \mathbb{R}$ is absolutely continuous inside $\tilde{I}$ if it is absolutely continuous on each compact subarc of $\tilde{I}$. (As a consequence, f' exists a.e. on $\tilde{I}$.)

Now let I be an interval or a complement of an interval in $\bar{\mathbb{R}}$. Let the arc $\tilde{I}$ be the image of I by the Cayley transform $T : \bar{\mathbb{R}} \to C$. We shall say that $f : I \to \mathbb{R}$ is absolutely continuous inside I if $f \circ T^{-1}$ is absolutely continuous inside $\tilde{I}$. (If I is a finite interval, this definition is equivalent to the usual definition of absolute continuity inside I.)

We say that a function f is increasing on an interval $I \subset \mathbb{R}$ if there exist $J \subset I$ such that the Lebesgue measure of $I \setminus J$ is zero and $f(x) \leq f(y)$ whenever $x, y \in J$, $x \leq y$. (This is a useful definition when f is defined only a.e. on I.) We define "decreasing" in a similar manner.

Moreover, we say that f is convex on an interval I if f is absolutely continuous inside I and f' is increasing on I.

We finally note that under Cayley transform (or its inverse), sets with positive capacity are transferred to sets with positive capacity.

The remainder of this paper is structured as follows. In Section 2, we present our main results and in Section 3 we present our proofs.

2. Main Results: The Circle and the Compactified Real Line. In this section we state our main results. We begin with our main results for the circle and compactified real line.

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2.1. Circle.

THEOREM 2.1. *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor\gamma, \delta\rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Assume that q is absolutely continuous inside I and*

$$\liminf_{\substack{x \to y \\ x \in I}} q(x) = q(y) \tag{2.1}$$

whenever y is an endpoint of I with $y \in I$. Let e^{ic} be any point which is not an interior point of $\widehat{I}$. Let $[\widehat{\alpha_1, \beta_1}], \ldots, [\widehat{\alpha_k, \beta_k}]$ be $k \geq 0$ arcs of C. Here, for all $1 \leq i \leq k$, $0 < \beta_i - \alpha_i \leq 2\pi$ and $(S_w \cup \widehat{I}) \subset [\widehat{\alpha_i, \beta_i}]$. Suppose further that I can be written as a disjoint union of $n \geq 1$ intervals $I_1, \ldots, I_n$ and for any fixed $1 \leq j \leq n$, either

$$e^{q(\theta)} \left[2 \sin\left(\frac{\theta - c}{2}\right) q'(\theta) - \cos\left(\frac{\theta - c}{2}\right)\right] \operatorname{sgn}\left(\sin\left(\frac{\theta - c}{2}\right)\right) \tag{2.2}$$

is increasing on I_j, or for some $1 \leq i \leq k$:

$$\sin\left(\frac{\theta - \alpha_i}{2}\right) \sin\left(\frac{\beta_i - \theta}{2}\right) q'(\theta) + \frac{1}{4} \sin\left(\theta - \frac{\alpha_i + \beta_i}{2}\right) \tag{2.3}$$

is increasing on I_j. Finally, we assume that

$$\limsup_{\theta \to \theta_0^-} q'(\theta) \leq \liminf_{\theta \to \theta_0^+} q'(\theta),$$

whenever θ_0 is an endpoint of I_j ($1 \leq j \leq n$) but not an endpoint of I. Then $S_w \cap \widehat{I}$ is an arc of C.

Here sgn denotes the signum function.

REMARK 2. *The choice of c is not important, see Remark 6 and the proof of Lemma 3.3. We also remark that if $\widehat{I}$ is the full circle, then one should check only condition (2.2) and ignore (2.3) which is a stronger assumption.*

Below we give a condition which guarantees that S_w is the full circle:

COROLLARY 2.2. *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I_1 := (\gamma_1, \gamma_1 + 2\pi)$ and $I_2 := (\gamma_2, \gamma_2 + 2\pi)$ where $e^{i\gamma_1} \neq e^{i\gamma_2}$. Assume that (2.2) is increasing on I_1 where $c := \gamma_1$, and (2.2) is increasing on I_2 where $c := \gamma_2$. Then $S_w = C$.*

Proof. By Theorem 2.1 $S_w \cap \widehat{I_1}$ is an arc of C. Let e^{ic} be an interior point of this arc, not identical to $e^{i\gamma_2}$. Choose ρ_1, ρ_2 such that $c < \rho_2 < \rho_1 < c + 2\pi$ and both of the arcs $(\widehat{c, \rho_1})$ and $(\widehat{\rho_2, c + 2\pi})$ contain only one of $e^{i\gamma_1}$ and $e^{i\gamma_2}$, say, $(\widehat{c, \rho_1})$ contains $e^{i\gamma_1}$ and $(\widehat{\rho_2, c + 2\pi})$ contains $e^{i\gamma_2}$.

Using the first observation of Remark 2, we see that (2.2) is increasing on (c, ρ_1) because (2.2) is increasing on (c, ρ_1) when at (2.2) c is replaced by γ_2. Similarly, (2.2) is increasing on $(\rho_2, c + 2\pi)$ because (2.2) is increasing on $(\rho_2, c + 2\pi)$ when at (2.2) c is replaced by γ_1. Thus (2.2) is increasing on $(c, c + 2\pi)$ and so $S_w = C$ by Theorem 2.1 and by the choice of c. ☐

Example. The following example illustrates the theorem.

Let $q(\theta) = \cos(5\theta)\sin(3\theta)$ defined on $\Sigma = [2.9, 3.18] \cup [3.95, 4]$. (We may define w to be zero outside Σ so that w is defined on C.) We claim that both $S_w \cap [\widehat{2.9, 3.18}]$ and $S_w \cap [\widehat{3.95, 4}]$ are arcs of C. (One of them may be an empty set.)

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Take $\alpha_1 = 2.9, \beta_1 = 4$ and $\alpha_2 = 3.95, \beta_2 = 3.18 + 2\pi$.

One can verify that (2.2) is satisfied on $[2.9, 3.17]$ but not on the whole $[2.9, 3.18]$. (At (2.2) c can be chosen to be any number such that e^{ic} is not an interior point of $\widehat{[2.9, 3.18]}$. Or, simply check the $(q')^2 + q'' + 1/4 \geq 0$ condition, see Remark 6.) Also, using α_1 and β_1 we see that (2.3) is not satisfied on the whole $[2.9, 3.18]$. However, (2.3) is satisfied on the subinterval $[3.17, 3.18]$ (see FIG. 2.1). So the combination of the (2.2) and (2.3) conditions implies that $S_w \cap \widehat{[2.9, 3.18]}$ is an arc.

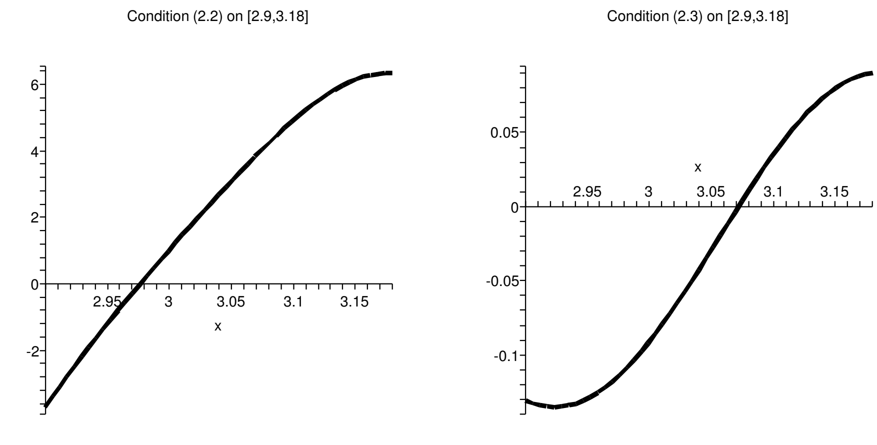

FIG. 2.1. *Conditions (2.2) and (2.3) on the interval I_1*

Using α_1 and β_1 on $[3.95, 4]$ is not helpful since (2.3) is a decreasing function there. Also, (2.2) is not satisfied on the whole $[3.95, 4]$. However, (2.3) is satisfied using α_2 and β_2 on the whole $[3.95, 4]$. Theorem 2.1 now implies that $S_w \cap \widehat{[3.95, 4]}$ is an arc (see FIG. 2.2). (We remark that α_2 and β_2 are not helpful on $[2.9, 3.18]$ since (2.3) is a decreasing function on $[3.17, 3.18]$.)

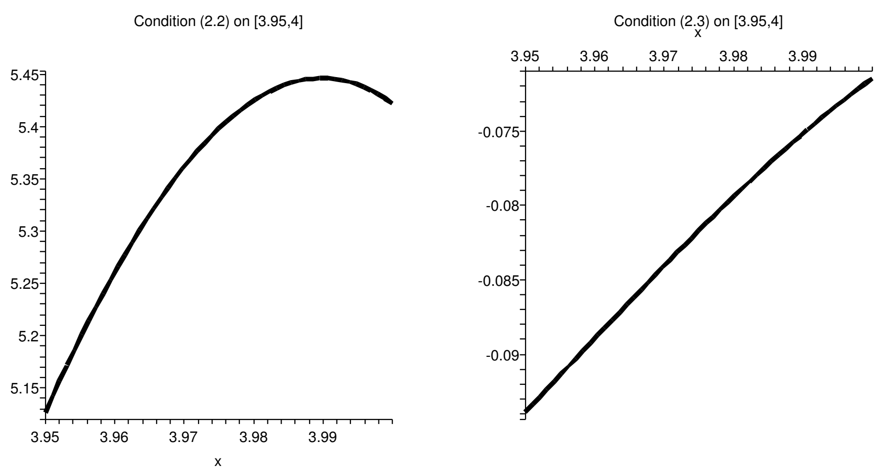

FIG. 2.2. *Conditions (2.2) and (2.3) on the interval I_2*

REMARK 3. *It is a natural question to ask what α_i and β_i numbers we should choose in*

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order that (2.3) is as weak as possible. In most cases the following statement is true:

Let $\widehat{[\alpha,\beta]}$ *and* $\widehat{[\alpha',\beta']}$ $(0<\beta-\alpha\le 2\pi,\ 0<\beta'-\alpha'\le 2\pi)$ *be two arcs of* C *such that* $S_w\subset\widehat{[\alpha,\beta]}\subset\widehat{[\alpha',\beta']}$. *Let* $\widehat{I}$ *be an arc contained in* $\widehat{[\alpha,\beta]}$. *If (2.2) or (2.3) is satisfied with* α',β' *then (2.2) or (2.3) is also satisfied with* α,β.

For example, this statement is true if $q''(\theta)$ *exists and the sets*

$$H:=\left\{\theta\in I: q'(\theta)>\frac{1}{2}\cot\left(\frac{\theta-\alpha'}{2}\right)\right\},\ H^*:=\left\{\theta\in I: q'(\theta)>\frac{1}{2}\cot\left(\frac{\theta-\beta'}{2}\right)\right\}$$

consist of finitely many intervals. (The proof of this is similar to the proof of [3, second remark].)

Theorem 2.1 can be effectively used when $w(z)$ *is identically zero on some arcs (that is,* Σ *is a subset of finitely many arcs). If* $w(z)$ *is zero on* $\widehat{[u_i,v_i]}$ $(0<v_i-u_i<2\pi)$, $i=1,\dots,k$, *then we may choose* $\widehat{[\alpha_i,\beta_i]}$ *to be* $\widehat{[v_i,u_i]}$ *in Theorem 2.1. This is consistent with the discussion above. For convenience we will state Theorem 2.3 in accordance with this remark.*

2.2. Compactified Real Line.

THEOREM 2.3. *For given* $k\in\mathbb{N}^+$ *let*

$$\Sigma:=\cup_{i=1}^k[A_i,B_i]\subset\bar{\mathbb{R}},\quad\text{where}$$
$$-\infty<A_1\le B_1<A_2\le B_2<\cdots<A_k\le B_k<+\infty.$$

Let $W=\exp(-Q)$ *be a weight on* Σ, $I\subset\Sigma$ *be an interval and assume that* Q *is absolutely continuous inside* I *and*

$$\liminf_{\substack{X\to Y\\ X\in I}} Q(X)=Q(Y), \tag{2.4}$$

whenever Y *is an endpoint of* I *with* $Y\in I$. *Assume further that* I *can be written as a disjoint union of intervals* $I_1,\dots,I_n$ *such that for any fixed* $1\le j\le n$ *either*

$$e^{Q(X)}\quad\text{is convex on } I_j,$$

or for some $1\le i\le k-1$

$$(X-B_i)(A_{i+1}-X)Q'(X)+X\quad\text{is decreasing on } I_j,$$

or

$$(X-A_1)(B_k-X)Q'(X)+X\quad\text{is increasing on } I_j. \tag{2.5}$$

Finally, we assume that

$$\limsup_{X\to X_0^-}Q'(X)\le\liminf_{X\to X_0^+}Q'(X),$$

whenever X_0 *is an endpoint of* I_j $(1\le j\le n)$ *but not an endpoint of* I. *Then* $S_W\cap I$ *is an interval.*

REMARK 4. *We remark that Theorem 2.3 is also valid when one interval, say,* $[A_k,B_k]$ *is an infinite interval or a complement of a finite interval. If* $A_k>B_k$ *(and, of course,* $B_k<A_1$*), then the conclusion of the theorem holds if (2.5) is replaced by the condition:*

$$(X-B_k)(A_1-X)Q'(X)+X\quad\text{is decreasing on } I_j. \tag{2.6}$$

If however $B_k = +\infty$, then (2.5) should be replaced by the condition:

$$(X - A_1)Q'(X) \quad \text{is increasing on } I_j.$$

Finally, if $A_1 = -\infty$ (and so $[A_1, B_1]$ is the infinite interval instead of $[A_k, B_k]$) then (2.5) should be replaced by the condition

$$(B_k - X)Q'(X) \quad \text{is increasing on } I_j.$$

At (2.6) and at Theorem 2.4 at (e) one can also consider an I which is a complement of a bounded interval. We leave the details for the reader.

Theorem 2.4 reveals to us the following remarkable connection between previously known conditions on Q. It also gives us a new condition (which is (e) below). As a consequence of Theorem 2.1 and 2.3 and Remark 4, we now have the following general result for the case when Σ is one real interval. See also [3]. Recall that for $A < B$ we define $[B, A] := (A, B)^c$.

THEOREM 2.4. *Let W be a weight on $\mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval. Assume that Q is absolutely continuous inside I and satisfies (2.4). Let $A \leq B$ be finite constants and suppose that either of the following conditions below hold:*

(a) $(X - A)(B - X)Q'(X) + X$ is increasing on $I \subset [A, B]$, $S_W \subset [A, B]$.
(b) $(X - A)Q'(X)$ is increasing on $I \subset [A, +\infty)$, $S_W \subset [A, +\infty)$.
(c) $(B - X)Q'(X)$ is increasing on $I \subset (-\infty, B]$, $S_W \subset (-\infty, B]$.
(d) $(X - A)^2Q'(X) - X$ is increasing on $I \subset \mathbb{R} \setminus \{A\}$,
(e) $(X - A)(B - X)Q'(X) + X$ is decreasing on $I \subset [B, A]$, $S_W \subset [B, A]$.
(f) Q is convex on I.
(g) $\exp(Q)$ is convex on I.

Then $S_W \cap I$ is an interval.

REMARK 5. *Theoretically one should ignore (d) and (f) since (g) is a weaker assumption than both of these. Nevertheless we included them here, because sometimes they are easier to check.*

Notice that (a) in Theorem 2.4 corresponds to the case of Theorem 2.1 when $\widehat{[\alpha, \beta]}$ is an arc of C disjoint of the point $x = 1$, (b) corresponds to the case when $\widehat{[\alpha, \beta]}$ is a proper subarc of C such that $\exp(i\beta) = 1$, (c) corresponds to the case when $\widehat{[\alpha, \beta]}$ is a proper subarc of C such that $\exp(i\alpha) = 1$, (d) corresponds to the case when $\widehat{[\alpha, \beta]}$ is the full circle C and a subcase of this is when $A = \infty$ (so $\alpha = 0$ and $\beta = 2\pi$) which corresponds to (f). The condition (e) corresponds to the case when $\widehat{[\alpha, \beta]}$ is a proper subarc of C which contains the point $x = 1$ inside the arc. Finally, (g) is the only condition which corresponds to (2.2) and not (2.3).

Note also that if we let $A = B$ then (e) leads to condition (d), since $(X - A)(A - X)Q'(X) + X$ is decreasing if and only if $(X - A)^2Q'(X) - X$ is increasing.

One may also combine the above conditions to create a weaker condition in the spirit of Theorem 2.1 and 2.3.

3. Proofs. In this section, we present the proofs of our results. We find it convenient to break down our proofs into several auxiliary lemmas. Our first lemma is

LEMMA 3.1. *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor\gamma, \delta\rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Let $0 < \beta - \alpha \leq 2\pi$ and assume $S_w \cup \widehat{I} \subset \widehat{[\alpha, \beta]}$. Suppose $q(\theta) := q(e^{i\theta})$ is absolutely continuous inside I and satisfies (2.1). Moreover, assume that*

$$\sin\left(\frac{\theta - \alpha}{2}\right) \sin\left(\frac{\beta - \theta}{2}\right) q'(\theta) + \frac{1}{4} \sin\left(\theta - \frac{\alpha + \beta}{2}\right) \tag{3.1}$$

is increasing on I. Then $S_w \cap \widehat{I}$ is an arc of C.

Proof. Let

$$A := -\cot\frac{\alpha}{2}, \quad B := -\cot\frac{\beta}{2}, \quad X := -\cot\frac{\theta}{2}.$$

First, let us assume that $\alpha, \beta \in (0, 2\pi)$. Thus, we may assume that $0 < \alpha \leq \gamma < \theta < \delta \leq \beta < 2\pi$ and $0 < \sin(\alpha/2), 0 < \sin(\beta/2)$. So $A \leq X \leq B$.

From (1.7), we have

$$Q'\left(-\cot\frac{\theta}{2}\right) = 2\sin^2\left(\frac{\theta}{2}\right)\left(q'(\theta) - \frac{1}{2}\cot\frac{\theta}{2}\right). \tag{3.2}$$

Thus,

$$\begin{aligned}
&(X-A)(B-X)Q'(X) + X \\
&= -\left(\cot\frac{\theta}{2} - \cot\frac{\alpha}{2}\right)\left(\cot\frac{\theta}{2} - \cot\frac{\beta}{2}\right) Q'\left(-\cot\frac{\theta}{2}\right) - \cot\frac{\theta}{2} \\
&= -\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\left(2q'(\theta) - \cot\frac{\theta}{2}\right) - \cot\frac{\theta}{2}.
\end{aligned}$$

Now we use the following identity which holds for any α, β, θ:

$$\cot\left(\frac{\theta}{2}\right)\left(\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} - 1\right) = \frac{\sin(\theta - \frac{\alpha+\beta}{2})}{2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})} - \frac{1}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right).$$

It follows that

$$\begin{aligned}
&(X-A)(B-X)Q'(X) + X \\
&= -2\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}q'(\theta) + \frac{\sin(\theta - \frac{\alpha+\beta}{2})}{2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})} - \frac{1}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right).
\end{aligned} \tag{3.3}$$

Because $0 < \sin(\alpha/2)$, $0 < \sin(\beta/2)$, the right hand side of (3.3) is increasing on I if and only if (3.1) holds. Thus, if (3.1) holds then $(X-A)(B-X)Q'(X)+X$ is increasing on $[-\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}]$. Now consider the corresponding equilibrium problem on $\bar{\mathbb{R}}$, as described in Section 1 and let S_W denote the corresponding equilibrium measure on $\bar{\mathbb{R}}$. Using [3, Theorem 7] we get that $S_W \cap [-\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}]$ is an interval. It follows that $S_w \cap \widehat{I}$ is an arc of C. This proves Lemma 3.1 for the case when $\alpha, \beta \in (0, 2\pi)$.

Now let $\alpha \leq 2\pi \leq \beta, \beta - \alpha < 2\pi$. Note that $0 \leq \sin(\alpha/2), 0 \geq \sin(\beta/2)$. We cannot apply [3, Theorem 7] because $B \leq A$ and X is outside $[B, A]$. However, we can use the observation that condition (3.1) is "rotation invariant."

Let $0 < \sigma$ be a number such that

$$0 < \alpha - \sigma =: \alpha^*, \quad \beta^* := \beta - \sigma < 2\pi,$$

and define

$$\gamma^* := \gamma - \sigma, \ \delta^* := \delta - \sigma,$$

$$q_2(\theta) := q(\theta + \sigma).$$

For $w_2 = \exp(-q_2)$ and the parameters $\alpha^*, \beta^*, \gamma^*, \delta^*$, we may apply the case we studied above to get that $S_{w_2} \cap \widehat{\lfloor \gamma^*, \delta^* \rceil}$ is an arc of C. But this new equilibrium problem is isomorphic to the original one in the sense that everything (including the support) is rotated by the angle σ. It follows that $S_w \cap \widehat{I}$ is an arc of C.

Finally, we need to establish the lemma for the case when $\widehat{I}$ is the full circle. So let $\beta - \alpha := 2\pi$. Using the rotation invariance we may assume that $\alpha = 0, \beta = 2\pi$. Condition (3.1) is now equivalent to

$$\sin^2\left(\frac{\theta}{2}\right) q'(\theta) - \frac{1}{4}\sin\theta \quad \text{is increasing.}$$

Using (3.2) we get

$$2\sin^2(\frac{\theta}{2}) q'(\theta) - \frac{1}{2}\sin\theta = Q'(-\cot\frac{\theta}{2}). \tag{3.4}$$

Thus, $Q'(-\cot\frac{\theta}{2})$ is increasing $(0 < \theta < 2\pi)$, that is, $Q'(X)$ is increasing, and so $Q(X)$ is convex. It is well known, see [17], that in this case the support S_W is an interval. (The proof works for our more general weight.) So S_w is again an arc. We have completed the proof Lemma 3.1. $\square$

As a corollary to Lemma 3.1, we have

LEMMA 3.2. *Let W be a weight on $\bar{\mathbb{R}}$, let J be a finite interval and suppose that Q is absolutely continuous inside J and satisfies condition (2.4). Let $A \leq B$ be finite constants with $J \subset [B, A]$, $S_W \subset [B, A]$ and assume that $(X - A)(B - X)Q'(X) + X$ is decreasing on J. Then $S_W \cap J$ is an interval.*

Proof. Recall that $[B, A] = (A, B)^c$, see Definition 1.1.

We may find $\alpha < \beta$ such that $B = -\cot(\alpha/2)$, $A = -\cot(\beta/2)$ and $\beta - \alpha \leq 2\pi$. Notice that $\sin(\alpha/2)\sin(\beta/2) < 0$ necessarily.

Let $J = \lfloor -\cot(\gamma/2), -\cot(\delta/2) \rceil$, where $\alpha \leq \gamma \leq \delta \leq \beta$ and so $\delta - \gamma \leq 2\pi$.

The left hand side of (3.3) is a decreasing function of X on J, and so the right hand side of (3.3) is a decreasing function of θ on $I := [\gamma, \delta]$. Multiply that right hand side by the negative constant $\sin(\alpha/2)\sin(\beta/2)$. In this way we get an increasing function of θ on $[\gamma, \delta]$. So condition (3.1) is satisfied and from Lemma 3.1, we deduce that $S_w \cap \widehat{[\gamma, \delta]}$ is an arc of C. This implies immediately that $S_W \cap J$ is an interval. Lemma 3.2 is proved. $\square$

Our final lemma is:

LEMMA 3.3. *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor \gamma, \delta \rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Suppose q is absolutely continuous inside I and satisfies (2.1). Let e^{ic} be any point which is not an interior point of $\widehat{I}$. If*

$$e^{q(\theta)}\left[2\sin\left(\frac{\theta - c}{2}\right) q'(\theta) - \cos\left(\frac{\theta - c}{2}\right)\right] \operatorname{sgn}\Big(\sin\left(\frac{\theta - c}{2}\right)\Big) \tag{3.5}$$

is increasing on I, then $S_w \cap \widehat{I}$ is an arc of C.

REMARK 6. *Whether (3.5) is increasing on I or not, it does not depend on the choice of c (as long as e^{ic} is not an interior point of $\widehat{I}$). The proof of this is given in the proof of Lemma 3.3. We also remark that if q is twice differentiable then condition (3.5) is easily seen to be equivalent to*

$$q'(\theta)^2 + q''(\theta) + \frac{1}{4} \geq 0, \quad \theta \in (\gamma, \delta)$$

(regardless of the value of c).

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We give the following example to Lemma 3.3. Let Σ be one or several closed arcs on the unit circle but not the full circle. Assume the weight w is zero on the complement of Σ. Let $e^{i\rho}$ be a point in the complement of Σ, and define

$$q(\theta) := q(e^{i\theta}) := \log|\sin\frac{\theta-\rho}{2}| + d,$$

where d is an arbitrary constant. The value of c is our choice so let $c := \rho$. Then (3.5) is increasing on the whole of Σ (in fact it is identically zero) and therefore S_w is a set of arcs. Moreover, each arc of Σ contains at most one arc of S_w.

Proof of Lemma 3.3. First we show that whether (3.5) is increasing on I or not, it does not depend on the choice of c. We do not assume the existence of q''.

Let $F(x)$ and $u(x)$ be two real functions on $(0,1)$ such that F is bounded and increasing, and u is non-negative and Lipschitz continuous. Then there exists $E \subset (0,1)$ of full measure such that

$$\int_a^b \Big(F(x)u(x)\Big)' dx \le (Fu)(b) - (Fu)(a) \ \text{ if } \ a,b \in E,\ a \le b.$$

This observation easily follows from Fatou's Lemma applied to the sequence of functions $[(Fu)(x+\epsilon_n) - (Fu)(x)]/\epsilon_n,\ \epsilon_n \to 0^+$.

Suppose e^{ic} and e^{ic_2} are not interior points of $\widehat{I}$. Denote now (3.5) by $F_c(\theta)$. Let $J \subset I$ such that J has full measure and $F_c(x) \le F_c(y)$ for all $x \le y$, $x, y \in J$. We define the domain of F_c and q' to be J. We have

$$e^{q(\theta)}q'(\theta) = \frac{F_c(\theta) + e^{q(\theta)}\Big(\cos\frac{\theta-c}{2}\Big)\operatorname{sgn}\Big(\sin\frac{\theta-c}{2}\Big)}{2\Big|\sin\frac{\theta-c}{2}\Big|}, \quad \theta \in J, \tag{3.6}$$

which shows that $e^q q'$ is differentiable a.e. on J. Simple calculation gives

$$0 \le F_c'(\theta) = 2\Big|\sin\frac{\theta-c}{2}\Big|\Big[(e^{q(\theta)}q'(\theta))' + \frac{1}{4}e^{q(\theta)}\Big] \ \text{ a.e. } \ \theta \in J. \tag{3.7}$$

Replace c by c_2 at the formula (3.5) and denote it by $F_{c_2}(\theta)$. Also, replace in that formula $e^q q'$ by the quotient at (3.6). Thus we see that with some $u(\theta)$, $v(\theta)$ functions $F_{c_2}(\theta) = F_c(\theta)u(\theta) + v(\theta)$ holds, where inside (γ,δ): the function u is non-negative and Lipschitz continuous, F_c is increasing and bounded, and v is absolutely continuous (since e^q is absolutely continuous inside I).

So, by the observation above, we have

$$\int_a^b (F_c u + v)' \le (F_c u)(b) + v(b) - (F_c u)(a) - v(a) = F_{c_2}(b) - F_{c_2}(a)$$

for a.e. $a, b \in I$, where $a \le b$. But this integral is non-negative, since $0 \le F_{c_2}'$ a.e. $\theta \in I$ follows from (3.7). Hence, $0 \le F_{c_2}(b) - F_{c_2}(a)$, i.e., F_{c_2} is increasing. And this is what we wanted to show.

We may assume that $c \le \gamma < \delta \le c + 2\pi$. Let us rotate now $\widehat{I}$ to a position such that the rotation takes e^{ic} to the point $x = 1$. Condition (3.5) will change accordingly to a new condition where now $c = 0$. (We denote the new rotated weight by $w = \exp(-q)$, too.) We now have to show that $S_w \cap \widehat{I}$ is an arc of C for the new S_w and new $\widehat{I}$. Once we have done that we simply rotate $\widehat{I}$ back to the original position and the proof is complete.

This argument shows that we can assume without loss of generality that $c = 0$ and $0 \le \gamma < \delta \le 2\pi$. Define

$$W\Big(\frac{1+x}{1-x}i\Big) := |1-x|w(x), \quad |x| = 1. \tag{3.8}$$

Using the arguments in Section 1.1, (3.8) may also be given as

$$W(X) := \frac{2w\Big(\frac{X-i}{X+i}\Big)}{\sqrt{1+X^2}}, \quad X \in \bar{\mathbb{R}}.$$

We define $Q(X)$ by $W(X) =: \exp(-Q(X))$. Since w is a weight on C, we know that W is a weight on $\bar{\mathbb{R}}$.

We now show that $e^{Q(X)}Q'(X)$ is increasing on

$$I^0 := \lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}\rceil.$$

Let $x = e^{i\theta}$. Note that from (1.7) we have

$$e^{Q(X)} = \frac{e^{q(\theta)}}{2|\sin\frac{\theta}{2}|}.$$

Using this and (3.2), for $\theta \in [0, 2\pi]$ we get

$$e^{Q(X)}Q'(X) = \frac{1}{2}e^{q(\theta)}(2\sin\frac{\theta}{2}q'(\theta) - \cos\frac{\theta}{2}). \tag{3.9}$$

Note that the right hand side of (3.9) is an increasing function of θ on I by assumption. Now we apply [3, Theorem 5], to conclude that $S_W \cap I^0$ is an interval. (Although this theorem is formulated for weights with $\lim_{|X|\to\infty} XW(X) = 0$, the argument in the proof may be applied word for word for the more general weights considered here. Naturally one should work with $U_W^{\mu_W}(X)$ in the proof.) Since $S_W \cap I^0$ is an interval, we conclude that $S_w \cap \widehat{I}$ is an arc of C. The proof of Lemma 3.3 is complete. $\square$

We are now ready to present the

Proof of Theorem 2.1. If $\widehat{I}$ is the full circle C, then it follows from the assumption that $e^{i\alpha_t} = e^{i\beta_t} = e^{ic} = e^{i\gamma}$ for all t. Now, if (2.3) is increasing on I_j, then (2.2) is also increasing on I_j, as one can see. (Choose γ to be zero and use (3.4), (3.9) and the fact that the convexity of Q implies the convexity of $\exp(Q)$.) So we can get the weakest assumption if we assume that (2.2) is increasing on the whole I, and we already know from Lemma 3.3 that Theorem 2.1 holds under such an assumption. Thus, let us assume that $\widehat{I}$ is not the full circle.

As in the proof of Lemma 3.1 and 3.3 we observe that the statement of Theorem 2.1 is "rotation invariant." So, we may assume that $\widehat{[\gamma, \delta]}$ does not contain the $x = 1$ point and $e^{i\alpha_t} \ne 1, e^{i\beta_t} \ne 1$ for any t. We can also assume that $c = 0$.

Let $X = -\cot(\theta/2)$, $A_i = -\cot(\alpha_i/2)$ $B_i = -\cot(\beta_i/2)$, and $Q(X)$ be defined by (1.5). Let I_j be given by

$$I_j = \lfloor \xi_j, \eta_j \rceil, \quad 0 < \eta_j - \xi_j < 2\pi,$$

and define

$$I_j^0 := \lfloor -\cot\frac{\xi_j}{2}, -\cot\frac{\eta_j}{2}\rceil, \quad I^0 := \lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}\rceil.$$

Note that I^0 is a finite subinterval of $\mathbb{R}$ and it is the disjoint union of the intervals I_j^0 ($j = 1, \ldots, n$). We assume that I_j^0 is numerated from left to right. Note also that $[A_i, B_i] \supset I_j^0$ (recall Definition 1.1).

By assumption, for any j ($1 \leq j \leq n$), we can find i ($1 \leq i \leq k$), such that either

$$e^{Q(X)} \quad \text{is convex on } \; I_j^0, \; \text{ or} \tag{3.10}$$

$$A_i < B_i \; \text{ and } \; (X - A_i)(B_i - X)Q'(X) + X \; \text{ is increasing on } \; I_j^0 \; \text{ or} \tag{3.11}$$

$$A_i \geq B_i \; \text{ and } \; (X - A_i)(B_i - X)Q'(X) + X \; \text{ is decreasing on } \; I_j^0. \tag{3.12}$$

((3.10) is coming from the argument in Lemma 3.3, (3.11) is from Lemma 3.1, and (3.12) is from Lemma 3.2.)

Let $E_1 := 1$. We can find positive constants $E_2, \ldots, E_n$ (uniquely) such that the following function f is a positive continuous function inside I^0. For $x \in I_j^0$ ($j = 1, \ldots, n$), let

$$f(x) := \begin{cases} E_k \exp(2Q(X)) & \text{if } (3.10) \text{ is satisfied on } I_j^0 \\ E_k(X - A_i)(B - X_i) & \text{if } (3.11) \text{ is satisfied on } I_j^0 \\ E_k(X - A_i)(X - B_i) & \text{if } (3.12) \text{ is satisfied on } I_j^0. \end{cases}$$

Let $W := \exp(-Q)$. We can use the argument in [3, Theorem 12] to deduce the result. For this purpose let $A = -\cot(\alpha/2)$ and $B = -\cot(\beta/2)$ be any two numbers such that $A < B$, $[A, B] \subset I^0$, $(A, B) \cap S_W = \emptyset$. Let $\mu_1 := \mu_w\Big|_{[(\alpha+\beta)/2, \widehat{(\alpha+\beta)/2+\pi}]}$, $\mu_2 := \mu - \mu_1$. Using $U_w^{\mu_w}(x) = U_w^{\mu_1}(x) + U_w^{\mu_2}(x)$ and the monotone convergence theorem it easily follows that $U_w^{\mu_w}(x)$ is absolutely continuous on $\widehat{[\alpha, \beta]}$, and so by (1.2) $U_W^{\mu_W}(X)$ is absolutely continuous on $[A, B]$. Also, as in [3] one can verify that

$$f(X)\frac{d}{dX}(U_W^{\mu_W}(X))$$

is strictly increasing on $[A, B]$. By [3, Lemma 4] we get that $S_W \cap [A, B]$ is an interval. It follows that $S_W \cap I^0$ is also an interval and $S_w \cap \widehat{I}$ is an arc of C. □

We conclude this section with

The Proof of Theorem 2.3 and Theorem 2.4. These follow easily using Theorem 2.1, Lemma 3.2 and the discussion in Section 1. □

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