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DUALITY OF q-POLYNOMIALS, ORTHOGONAL ON COUNTABLE SETS OF POINTS*

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Abstract. We review properties of q-orthogonal polynomials, related to their orthogonality, duality and connection with the theory of symmetric (self-adjoint) operators, represented by a Jacobi matrix. In particular, we show how one can naturally interpret the duality of families of polynomials, orthogonal on countable sets of points. In order to obtain orthogonality relations for dual sets of polynomials, we propose to use two symmetric (self-adjoint) operators, representable (in some distinct bases) by Jacobi matrices. To illustrate applications of this approach, we apply it to several pairs of dual families of q-polynomials, orthogonal on countable sets, from the q-Askey scheme. For each such pair, the corresponding operators, representable by Jacobi matrices, are explicitly given. These operators are employed in order to find explicitly sets of points, on which the polynomials are orthogonal, and orthogonality relations for them.

Key words. q-orthogonal polynomials, duality, Jacobi matrix, orthogonality relations

AMS subject classifications. 33D80; 33D45; 17B37

"We mathematicians are particularly fond of duality theorems; translating mathematical statements from one category to another often gives us new and unexpected insight", M.Harris, "Postmodern at an Early Age", Notices of the American Mathematical Society, Vol.50, No.7, p.792, 2003.

1. Introduction. It is well known that each family $p_n(x)$, $n = 0, 1, 2, \cdots$, of orthogonal polynomials in one variable corresponds to the determinate or indeterminate moment problem. If a polynomial family corresponds to the determinate moment problem, then there exists only one positive orthogonality measure μ for these polynomials and they constitute a complete orthogonal set in the Hilbert space $L^2(\mu)$. If a family corresponds to the indeterminate moment problem, then there exists infinitely many orthogonality measures μ for these polynomials and these measures are divided into two parts: extremal measures and non-extremal measures. If a measure μ is extremal, then the corresponding set of polynomials constitute a complete orthogonal set in the Hilbert space $L^2(\mu)$. If a measure μ is not extremal, then the corresponding family of polynomials is not complete in the Hilbert space $L^2(\mu)$ (see [31]).

It is also well known that there exists a close relation of the theory of orthogonal polynomials with the theory of symmetric (self-adjoint) operators, representable by a Jacobi matrix. The point is that with each family of orthogonal polynomials one can associate a closed symmetric (or self-adjoint) operator A, representable by a Jacobi matrix. If the corresponding moment problem is indeterminate, then the operator A is not self-adjoint and it has infinitely many self-adjoint extensions. If the operator A has a physical meaning, then these self-adjoint extensions are especially important. These extensions correspond to extremal orthogonality measures for the same set of polynomials and can be constructed by means of these measures (see, for example, [15], Chapter VII, and [32]). If the family of polynomials corresponds to the determinate moment problem, then the corresponding operator A is self-adjoint and its spectrum is determined by an orthogonality relation for the polynomials. Moreover, the spectral measure for the operator A is constructed by means of the orthogonality measure for the corresponding polynomials (see [15], Chapter VII).

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In section 2, we briefly review the relations between the theory of orthogonal polynomials, the theory of operators, representable by a Jacobi matrix, and the theory of moment problem. This information is basic for the exposition in further sections. In section 2, we also discuss how one can naturally extend the conventional notion of duality to sets of polynomials, orthogonal on countable sets of points.

In order to find orthogonality measures for dual sets of polynomials, we use two symmetric (or self-adjoint) operators, representable (with respect to different bases) by Jacobi matrices. This approach is applied to several sets of dual q-orthogonal polynomials from the Askey scheme.

Pairs of operators (A, I), employed for studying some sets of q-orthogonal polynomials and their duals, belong to the discrete series representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$ (see, for example, [6] and [11]). However, in order to facilitate ease of comprehending to a larger number of readers we have not exploited this deep algebraic fact; that is, we exhibit explicit forms of these operators without using the representation theory of the quantum algebra $U_q(\mathrm{su}_{1,1})$. These pairs of operators are, in fact, a generalization of Leonard pairs, introduced by P. Terwilliger [35] (for the definition and references see section 3).

When one considers dual sets of q-polynomials, orthogonal on countable sets of points, then one member of these sets corresponds to the determinate moment problem and another to the indeterminate moment problem. One of the two operators (A, I) (that is, the operator A) for a given dual pair of sets of q-orthogonal polynomials corresponds to a three-term recurrence relation for the set of polynomials, which corresponds to the determinate moment problem. This operator is bounded and self-adjoint; moreover, it has the discrete spectrum. We diagonalize this self-adjoint bounded operator and find its spectrum with the aid of the second operator I, which corresponds to a q-difference equation for the same set of polynomials. An explicit form of all eigenfunctions for the operator A is found for each dual set of polynomials, considered by us. They are expressed in terms of q-polynomials, which belong to the set, associated with the determinate moment problem. Since the spectrum of A is simple, its eigenfunctions form an orthogonal basis in the Hilbert space. One can normalize this basis. This normalization is effected by means of the second operator I from the corresponding pair. As a result of this normalization, two orthonormal bases in the Hilbert space emerge: the canonical (or the initial) basis and the basis of eigenfunctions of the operator A. They are interrelated by a unitary matrix U, whose entries u_{mn} are explicitly expressed in terms of polynomials $P_m(x)$, corresponding to the determinate moment problem. Since the matrix U is unitary (and in fact it is real in our case), there are two orthogonality relations for its elements, namely

$$\sum_n u_{mn}\, u_{m'n} = \delta_{mm'}, \quad \sum_m u_{mn}\, u_{mn'} = \delta_{nn'}. \tag{1.1}$$

The first relation expresses the orthogonality relation for the polynomials $P_m(x)$, which correspond to the determinate moment problem. So, the orthogonality of U yields an algebraic proof of orthogonality relation for these polynomials. In order to interpret the second relation, we consider the polynomials $P_m(x_n)$ (where $\{x_n\}$ is the set of points, on which the polynomials are orthogonal) as functions of m. In this way one obtains one or two sets of orthogonal functions, which are expressed in terms of a dual set of q-orthogonal polynomials (which corresponds to the indeterminate moment problem). The second relation in (1.1) leads to the orthogonality relations for these dual q-orthogonal polynomials.

Since this set of q-orthogonal polynomials corresponds to the indeterminate moment problem, there are infinitely many orthogonality relations. Using the pair of operators (A, I) and the notion of duality, one is able to find only one orthogonality relation (which is dual to the orthogonality relation for the corresponding set of polynomials, associated with the determinate moment problem). Sometimes a measure, which corresponds to this orthogonality relation, is extremal and sometimes it is not extremal. It depends on a concrete pair of dual sets of polynomials.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions and the standard q-analysis (see, for example, [21] and [3]). We shall also use the well-known shorthand notation $(a_1, \cdots, a_k; q)_n := (a_1; q)_n \cdots (a_k; q)_n$.

2. Orthogonality measures and duality.

2.1. Orthogonal polynomials, Jacobi matrices and the moment problem. Orthogonal polynomials are closely related to operators represented by a Jacobi matrices. In what follows we shall use only symmetric Jacobi matrices and the word "symmetric" will be often omitted. By a symmetric Jacobi matrix we mean a (finite or infinite) symmetric matrix of the form

$$M = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{2.1}$$

We assume below that all $a_i \neq 0$, $i = 0, 1, 2, \cdots$; then a_i are real. Let L be a closed symmetric operator on a Hilbert space $\mathcal{H}$, representable by a Jacobi matrix M. Then there exists an orthonormal basis e_n, $n = 0, 1, 2, \cdots$, in $\mathcal{H}$, such that

$$Le_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1},$$

where $e_{-1} \equiv 0$. Let $|x\rangle = \sum_{n=0}^{\infty} p_n(x) e_n$ be an eigenvector[1] of L with an eigenvalue x, that is, $L|x\rangle = x|x\rangle$. Then

$$L|x\rangle = \sum_{n=0}^{\infty} [p_n(x) a_n e_{n+1} + p_n(x) b_n e_n + p_n(x) a_{n-1} e_{n-1}] = x \sum_{n=0}^{\infty} p_n(x) e_n.$$

Equating coefficients of the vector e_n, one comes to a recurrence relation for the coefficients $p_n(x)$:

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x). \tag{2.2}$$

Since $p_{-1}(x) = 0$, by setting $p_0(x) \equiv 1$ we see that $p_1(x) = a_0 x - b_0/a_0$. Similarly we can find uniquely $p_n(x)$, $n = 2, 3, \cdots$. Thus, the relation (2.2) completely determines the coefficients $p_n(x)$. Moreover, the recursive computation of $p_n(x)$ shows that these coefficients $p_n(x)$ are polynomials in x of degree n. Since the coefficients a_n and b_n are real (because the matrix M is symmetric), all coefficients of the polynomials $p_n(x)$ themselves are real.

Well-known Favard's characterization theorem for polynomials $P_n(x)$, $n = 0, 1, 2, \cdots$, of degree n states that if these polynomials satisfy a recurrence relation

$$A_n\, P_{n+1}(x) + B_n\, P_n(x) + C_n\, P_{n-1}(x) = x\, P_n(x)$$

[1] Observe that eigenvectors of L may belong to either the Hilbert space $\mathcal{H}$ or to some extension of $\mathcal{H}$. (For example, if $\mathcal{H} = L^2(-\infty, \infty)$ and in place of L we have the operator d/dx, then the functions e^{ixp}, which do not belong to $L^2(-\infty, \infty)$, are eigenfunctions of d/dx.) Below we act freely with eigenvectors, which do not belong to $\mathcal{H}$, but this can be easily made mathematically strict.

and the conditions $A_nC_{n+1} > 0$ are satisfied, then these polynomials are orthogonal with respect to some positive measure. It is clear that the conditions of Favard's theorem are satisfied for the polynomials $p_n(x)$ because in this case the requirements simply reduce to inequalities $a_n^2 > 0$ for $n = 0, 1, 2, \cdots$. This means that the polynomials $p_n(x)$ from (2.2) are orthogonal with respect to some positive measure $\mu(x)$. It is known that orthogonal polynomials admit orthogonality with respect to either unique positive measure or with respect to infinitely many positive measures.

The polynomials $p_n(x)$ are very important for studying properties of the closed symmetric operator L. Namely, the following statements are true (see, for example, [15] and [32]):

I. Let the polynomials $p_n(x)$ are orthogonal with respect to a unique orthogonality measure μ,

$$\int p_m(x)p_n(x)d\mu(x) = \delta_{mn},$$

where the integration is performed over some subset (possibly discrete) of $\mathbb{R}$, then the closed operator L is self-adjoint. Moreover, the spectrum of the operator L is simple and coincides with the set, on which the polynomials $p_n(x)$ are orthogonal (recall that we assume that all numbers a_n are non-vanishing). The measure $\mu(x)$ determines the spectral measure for the operator L (for details see [15], Chapter VII).

II. Let the polynomials $p_n(x)$ are orthogonal with respect to infinitely many different orthogonality measures μ. Then the closed symmetric operator L is not self-adjoint and has deficiency indices (1, 1), that is, it has infinitely many (in fact, one-parameter family of) self-adjoint extensions. It is known that among orthogonality measures, with respect to which the polynomials are orthogonal, there are so-called extremal measures (that is, such measures that a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert space L^2 with respect to the corresponding measure; see subsection 2.3 below). These measures uniquely determine self-adjoint extensions of the symmetric operator L. There exists one-to-one correspondence between essentially distinct extremal orthogonality measures and self-adjoint extensions of the operator L. The extremal orthogonality measures determine spectra of the corresponding self-adjoint extensions.

The inverse statements are also true:

I$'$. Let the operator L be self-adjoint. Then the corresponding polynomials $p_n(x)$ are orthogonal with respect to a unique orthogonality measure μ,

$$\int p_m(x)p_n(x)d\mu(x) = \delta_{mn},$$

where the integral is taken over some subset (possibly discrete) of $\mathbb{R}$, which coincides with the spectrum of L. Moreover, a measure μ is uniquely determined by a spectral measure for the operator L (for details see [15], Chapter VII).

II$'$. Let the closed symmetric operator L be not self-adjoint. Since it is representable by a Jacobi matrix (2.1) with $a_n \neq 0$, $n = 0, 1, 2, \cdots$, it admits one-parameter family of self-adjoint extensions (see [15], Chapter VII). Then the polynomials $p_n(x)$ are orthogonal with respect to infinitely many orthogonality measures μ. Moreover, spectral measures of self-adjoint extensions of L determine extremal orthogonality measures for the polynomials $\{p_n(x)\}$ (and a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert spaces $L^2(\mu)$ with respect to the corresponding extremal measures μ).

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On the other hand, with the orthogonal polynomials $p_n(x)$, $n = 0, 1, 2, \cdots$, the classical moment problem is associated (see [31] and [12]). Namely, with these polynomials (that is, with the coefficients a_n and b_n in the corresponding recurrence relation) real numbers c_n, $n = 0, 1, 2, \cdots$, are associated, which determine the corresponding classical moment problem. (The numbers c_n are uniquely determined by a_n and b_n.) The definition of the classical moment problem consists in the following. Let a set of real numbers c_n, $n = 0, 1, 2, \cdots$, be given. We are looking for a positive measure $\mu(x)$, such that

$$\int x^n d\mu(x) = c_n, \quad n = 0, 1, 2, \cdots , \tag{2.3}$$

where the integration is taken over $\mathbb{R}$. (In this case we deal with the *Hamburger moment problem*.) There are two principal questions in the theory of moment problem:

(i) Does there exist a measure $\mu(x)$, such that relations (2.3) are satisfied?
(ii) If such a measure exists, is it determined uniquely?

The answer to the first question is positive, if the numbers c_n, $n = 0, 1, 2, \cdots$, are those, which correspond to a family of orthogonal polynomials. Moreover, a measure $\mu(x)$ then coincides with the measure, with respect to which these polynomials are orthogonal.

If a measure μ in (2.3) is determined uniquely, then we say that we deal with *the determinate moment problem*. In particular, it is the case when the measure μ is supported on a bounded set. If a measure, with respect to which relations (2.3) hold, is not unique, then we say that we deal with *the indeterminate moment problem*. In this case there exist infinitely many measures $\mu(x)$ for which (2.3) take place. Then the corresponding polynomials are orthogonal with respect to all these measures and the corresponding symmetric operator L is not self-adjoint. In this case the set of solutions of the moment problem for the numbers $\{c_n\}$ coincides with the set of orthogonality measures for the corresponding polynomials $\{p_n(x)\}$.

Observe that not each set of real numbers c_n, $n = 0, 1, 2, \cdots$, is associated with a set of orthogonal polynomials. In other words, there are sets of real numbers c_n, $n = 0, 1, 2, \cdots$, such that the corresponding moment problem does not have a solution, that is, there is no positive measure μ, for which the relations (2.3) are true. But if for some set of real numbers c_n, $n = 0, 1, 2, \cdots$, the moment problem (2.3) has a solution μ, then this set corresponds to some set of polynomials $p_n(x)$, $n = 0, 1, 2, \cdots$, which are orthogonal with respect to this measure μ. There exist criteria indicating when for a given set of real numbers c_n, $n = 0, 1, 2, \cdots$, the moment problem (2.3) has a solution (see, for example, [31]). Moreover, there exist procedures, which associate a collection of orthogonal polynomials to a set of real numbers c_n, $n = 0, 1, 2, \cdots$, for which the moment problem (2.3) has a solution (see, [31]).

Thus, we see that the following three theories are closely related:

(i) the theory of symmetric operators L, representable by a Jacobi matrix;
(ii) the theory of orthogonal polynomials in one variable;
(iii) the theory of classical moment problem.

2.2. Extremal orthogonality measures. To a set of orthogonal polynomials $p_n(x)$, $n = 0, 1, 2, \cdots$, associated with an indeterminate moment problem (2.3), there correspond four entire functions $A(z)$, $B(z)$, $C(z)$, $D(z)$, which are related to appropriate orthogonality measures μ for the polynomials by the formula

$$F(z) \equiv \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)} = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z - t} \tag{2.4}$$

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(see, for example, [31]), where $\sigma(z)$ is any analytic function. Moreover, to each analytic function $\sigma(z)$ (including cases of constant $\sigma(z)$ and $\sigma(z)=\pm\infty$) there corresponds a single orthogonality measure $\mu(t)\equiv\mu_\sigma(t)$ and, conversely, to each orthogonality measure μ there corresponds an analytic function σ such that formula (2.4) holds. There exists the Stieltjes inversion formula, which converts the formula (2.4). It has the form

$$[\mu(t_1+0)+\mu(t_1-0)]-[\mu(t_0+0)+\mu(t_0-0)]$$

$$=\lim_{\varepsilon\to+0}\left(-\frac{1}{\pi\mathrm{i}}\int_{t_0}^{t_1}[F(t+\mathrm{i}\varepsilon)-F(t-\mathrm{i}\varepsilon)]dt\right).$$

Thus, orthogonality measures for a given set of polynomials $p_n(x)$, $n=0,1,2,\cdots$, in principle, can be found. However, it is very difficult to evaluate the functions $A(z)$, $B(z)$, $C(z)$, $D(z)$. In [23] they are evaluated for particular example of polynomials, namely, for the q^{-1}-continuous Hermite polynomials $h_n(x|q)$. So, as a rule, for the derivation of orthogonality measures other methods are used.

The measures $\mu_\sigma(t)$, corresponding to constants σ (including $\sigma=\pm\infty$), are called *extremal measures* (some authors, following the book [12], call these measures N-extremal). All other orthogonality measures are not extremal.

The importance of extremal measures is explained by Riesz's theorem. Let us suppose that a set of polynomials $p_n(x)$, $n=0,1,2,\cdots$, associated with the indeterminate moment problem, is orthogonal with respect to a positive measure μ (that is, μ is a solution of the moment problem (2.3)). Let $L^2(\mu)$ be the Hilbert space of square integrable functions with respect to the measure μ. Evidently, the polynomials $p_n(x)$ belong to the space $L^2(\mu)$. Riesz's theorem states the following:

THEOREM 2.1. *The set of polynomials $p_n(x)$, $n=0,1,2,\cdots$, is complete in the Hilbert space $L^2(\mu)$ (that is, they form a basis in this Hilbert space) if and only if the measure μ is extremal.*

Note that if a set of polynomials $p_n(x)$, $n=0,1,2,\cdots$, corresponds to a determinate moment problem and μ is an orthogonality measure for them, then this set of polynomials is also complete in the Hilbert space $L^2(\mu)$.

In particular, Riesz's theorem 2.1 is often used in order to determine whether a certain orthogonality measure is extremal or not. Namely, if we know that a given set of orthogonal polynomials, corresponding to an indeterminate moment problem, is not complete in the Hilbert space $L^2(\mu)$, where μ is an orthogonality measure, then this measure is not extremal.

Note that for applications in physics and in functional analysis it is of interest to have extremal orthogonality measures. If an orthogonality measure μ is not extremal, then it is important to find a system of orthogonal functions $\{f_m(x)\}$, which together with a given set of polynomials constitute a complete set of orthogonal functions (that is, a basis in the Hilbert space $L^2(\mu)$). Sometimes, it is possible to find such systems of functions (see, for example, [19]).

Extremal orthogonality measures have many interesting properties [31]:

(a) If $\mu_\sigma(x)$ is an extremal measure, associated (according to formula (2.4)) with a number σ, then $\mu_\sigma(x)$ is a step function. Its spectrum (that is, the set on which the corresponding polynomials $p_n(x)$, $n=0,1,2,\cdots$, are orthogonal) coincides with the set of zeros of the denominator $B(z)-\sigma D(z)$ in (2.4). The mass, concentrated at a spectral point x_j (that is, a jump of $\mu_\sigma(x)$ at the point x_j), is equal to $(\sum_{n=0}^\infty |p_n(x_j)|^2)^{-1}$.

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(b) Spectra of extremal measures are real and simple. This means that the corresponding self-adjoint operators, which are self-adjoint extensions of the operator L, have simple spectra, that is, all spectral points are of multiplicity 1.

(c) Spectral points of two different extremal measures $\mu_\sigma(x)$ and $\mu_{\sigma'}(x)$ are mutually separated.

(d) For a given real number x_0, always exists a (unique) real value σ, such that the measure $\mu_\sigma(x)$ has x_0 as its spectral point. The points of the spectrum of $\mu_\sigma(x)$ are analytic monotonic functions of σ.

It is difficult to find all extremal orthogonality measures for a given set of orthogonal polynomials (that is, self-adjoint extensions of a corresponding closed symmetric operator). As far as we know, at the present time they are known only for one family of polynomials, which correspond to indeterminate moment problem. They are the q^{-1}-continuous Hermite polynomials $h_n(x|q)$ (see [23]).

If extremal measures μ_σ are known then by multiplying μ_σ by a suitable factor (depending on σ) and integrating it with respect to σ, one can obtain infinitely many continuous orthogonality measures (which are not extremal).

2.3. Dual sets of orthogonal polynomials. A notion of duality for two families of polynomials, orthogonal on finite sets of points, is well known. Namely, let $p_n(x)$, $n = 0, 1, 2, \cdots, N$, be orthogonal polynomials with orthogonality relation

$$\sum_{m=0}^{N} p_n(x_m) p_{n'}(x_m) w_m = v_n^{-1} \delta_{nn'} \sum_{s=0}^{N} w_s, \tag{2.5}$$

where $n, n' = 0, 1, 2, \cdots, N$, $w_s > 0$ is a jump of the orthogonality measure in the point x_s, and

$$v_n = \prod_{k=0}^{n} (a_{k-1}/c_k)$$

(a_k and c_n are coefficients in the three-term recurrence relation $xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$ for the polynomials $p_n(x)$). Then the dual orthogonality relation is of the form

$$\sum_{n=0}^{N} p_n(x_m) p_n(x_{m'}) v_n = w_m^{-1} \delta_{mm'} \sum_{s=0}^{N} w_s, \tag{2.6}$$

where $m, m' = 0, 1, 2, \cdots, N$ (see, for example, [21], Chapter 7). If one considers the $p_n(x_m)$ as functions of n, in many cases these functions turn out to be either polynomials in n or polynomials in $\nu(n)$, where $\nu(n)$ is some function of n. Then the polynomials $P_m(n) := p_n(x_m)$, $m = 0, 1, 2, \cdots, N$ (respectively, $P_m(\nu(n)) := p_n(x_m)$, $m = 0, 1, 2, \cdots, N$) are orthogonal polynomials of n (respectively of $\nu(n)$), for which (2.6) is an orthogonality relation. The polynomials $P_m(\nu(n))$, $m = 0, 1, 2, \cdots, N$, are called *dual* polynomials with respect to the $p_n(x_m)$, $n = 0, 1, 2, \cdots, N$. If the dual polynomials $\{P_m(\nu(n))\}$ coincide with the $\{p_n(m)\}$, then the polynomials $\{p_n(m)\}$ are called *self-dual*. For instance, Racah polynomials and q-Racah polynomials both represent families of self-dual polynomials.

It is not obvious how to extend the notion of duality to polynomials, orthogonal on countable sets of points. In the case of polynomials, orthogonal on a finite set of points, the orthogonality (2.6) readily follows from the orthogonality (2.5). Namely, the orthogonality (2.5) means that the real $(N+1)\times(N+1)$ matrix $(a_{mn})_{m,n=0}^{N}$ with matrix elements

$$a_{mn} = c_{mn} p_n(x_m), \quad c_{mn} = \left[w_m v_n / \sum_{s=0}^{N} w_s \right]^{1/2},$$

is orthogonal. Orthogonality of its columns is equivalent to the relation (2.5). Orthogonality by rows for the matrix $(a_{mn})_{m,n=0}^{N}$ yields the relation (2.6).

In the case, when we have orthogonality of polynomials on a countable set of points a similar conclusion can be false. Let $p_n(x)$, $n = 0, 1, 2, \cdots$, be a set of orthogonal polynomials with orthogonality relation

$$\sum_{m=0}^{\infty} p_n(x_m)\, p_{n'}(x_m)\, w_m = h_n\, \delta_{nn'}, \tag{2.7}$$

where $n, n' = 0, 1, 2, \cdots$ and h_n are some constants. Again, one may consider $p_n(x_m)$ as functions of n. We are interested in the cases when these functions are polynomials either in n or in some $\nu(n)$. So the question arises: When the dual relation to (2.7), namely,

$$\sum_{n=0}^{\infty} p_n(x_m)\, p_n(x_{m'}))\, h_n^{-1} = w_m^{-1}\, \delta_{mm'}, \tag{2.8}$$

is an orthogonality relation for the dual polynomials $P_m(\nu(n)) := p_n(x_m)$, $m = 0, 1, 2, \cdots$? It follows from Riesz's theorem 2.1 that *this is the case, when the orthogonality measure in* (2.7) *corresponds to determinate moment problem or when this measure corresponds to indeterminate moment problem and it is extremal.* Namely, in both these cases the matrix $(a_{mn})_{m,n=0}^{\infty}$ with $a_{mn} = (h_n^{-1} w_m)^{1/2} p_n(x_m)$ is orthogonal, that is,

$$\sum_m a_{mn}\, a_{mn'} = \delta_{nn'}, \quad \sum_n a_{mn}\, a_{m'n} = \delta_{mm'}.$$

It is natural to call the polynomials $P_m(\nu(n))$ *dual* to the polynomials $p_n(m)$. The orthogonality relation for them is

$$\sum_{n=0}^{\infty} P_m(\nu(n))\, P_{m'}(\nu(n))\, h_n^{-1} = w_m^{-1}\, \delta_{mm'}.$$

However, very often a function $\nu(n)$, such that $P_m(\nu(n)) := p_n(x_m)$, $m = 0, 1, 2, \cdots$, are polynomials in $\nu(n)$, does not exist. Nevertheless, sometimes it turns out that there are some m-independent b_n, $n = 0, 1, 2, \cdots$, such that $P_m(\nu(n)) := b_n\, p_n(x_m)$, $m = 0, 1, 2, \cdots$, are polynomials in $\nu(n)$ for an appropriate function $\nu(n)$. When the orthogonality measure in (2.7) corresponds to determinate moment problem or when this measure corresponds to indeterminate moment problem and it is extremal, then we have the orthogonality relation (2.8) for the functions $\hat{P}_m(\nu(n)) := p_n(x_m)$, which is equivalent to the orthogonality relation

$$\sum_{n=0}^{\infty} P_m(\nu(n))\, P_{m'}(\nu(n))\, b_n^{-2}\, h_n^{-1} = w_m^{-1}\, \delta_{mm'}$$

for the polynomials $P_m(\nu(n))$. In this case it is also natural to call the polynomials $P_m(\nu(n)) = b_n\, p_n(x_m)$ *dual* to the orthogonal polynomials $p_n(m)$.

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The situation can be sometimes more complicated. Namely, the orthogonality relation for polynomials with a weight function, supported on a countable set of points, may be of the form (for instance, for the big q-Jacobi polynomials)

$$\sum_{m=0}^{\infty} p_n(x_m)\, p_{n'}(x_m) w_m + \sum_{m=0}^{\infty} p_n(y_m)\, p_{n'}(y_m)\, w'_m = h_n\, \delta_{nn'}. \tag{2.9}$$

Let $b_n, b'_n, \nu(n)$ and $\nu'(n)$ be such functions of n that $P_m(\nu(n)) := b_n\, p_n(x_m)$ and $P'_m(\nu'(n))$ $:= b'_n\, p_n(y_m)$ are polynomials in $\nu(n)$ and $\nu'(n)$, respectively. When the orthogonality measure in (2.9) corresponds to determinate moment problem or when this measure corresponds to indeterminate moment problem and it is extremal, then the matrix $\begin{pmatrix} (a_{mn})_{m,n=0}^{\infty} \\ (a'_{mn})_{m,n=0}^{\infty} \end{pmatrix}$, with two infinite matrices (placed one over another) with matrix elements $a_{mn} = (h_n^{-1} w_n)^{1/2} p_n(x_m)$ and $a'_{mn} = (h_n^{-1} w'_n)^{1/2} p_n(y_m)$, is orthogonal, that is,

$$\sum_m a_{mn}\, a_{mn'} + \sum_m a'_{mn}\, a'_{mn'} = \delta_{nn'},$$

$$\sum_n a_{mn}\, a_{m'n} = \delta_{mm'}, \quad \sum_n a'_{mn}\, a'_{m'n} = \delta_{mm'}, \quad \sum_n a_{mn}\, a'_{m'n} = 0.$$

Orthogonality of columns of this matrix gives the orthogonality relation (2.9). The orthogonality of rows gives orthogonality of the polynomials $P_m(\nu'(n))$ and $P'_m(\nu'(n))$:

$$\sum_{n=0}^{\infty} P_m(\nu(n))\, P_{m'}(\nu(n))\, b_n^{-2}\, h_n^{-1} = w_m^{-1}\, \delta_{mm'},$$

$$\sum_{n=0}^{\infty} P'_m(\nu'(n))\, P'_{m'}(\nu'(n))\, {b'}_n^{-2}\, h_n^{-1} = {w'}_m^{-1}\, \delta_{mm'},$$

$$\sum_{n=0}^{\infty} P_m(\nu(n))\, P'_{m'}(\nu'(n))\, b_n^{-1}\, {b'}_n^{-1}\, h_n^{-1} = 0.$$

In this case both sets of the polynomials $P_m(\nu'(n))$ and $P'_m(\nu'(n))$ are regarded on a par as *duals* to the set of orthogonal polynomials $p_n(x)$. (For the big q-Jacobi polynomials, these two dual sets turn out to be polynomials of the same type, but with different values of parameters; see section 4.) If the orthogonality relation (2.9) would contain r terms, then we had r dual sets of functions.

Thus, if we have a set of polynomials $\{p_n(x)\}$, orthogonal on a countable set of points, and they correspond to a determinate moment problem or to an indeterminate moment problem and the orthogonality measure is extremal, then one can find the corresponding orthogonality measure for dual set of polynomials, if they exist.

The main goal of this review is to discuss a method of constructing orthogonality measures for a given set of polynomials and their duals in a straightforward manner. This method is based on the use of two closed symmetric (self-adjoint) operators, representable (in some bases) by Jacobi matrices. In the following sections, we shall illustrate how this method works by considering families of q-orthogonal polynomials from the Askey scheme.

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Let us emphasize that there are already known theorems on dual orthogonality properties of polynomials, whose weight functions are supported on an infinite set of discrete points (see, for example, [20], [34], [16] and [24]). But it is essential that in most cases (especially in the cases of q-polynomials) dual objects are represented by orthogonal functions. Therefore, one still needs to make one step further in order to single out an appropriate family of dual polynomials from these functions (in those cases when it turns out to be possible). We show that this step can be made by choosing the b_n for the dual polynomials $P_m(\nu(n)) := b_n\, p_n(x_m)$. Besides, when one considers some dual set with respect to a given family of orthogonal polynomials, it is also necessary to investigate the problem of completeness for this dual object. In our approach, based on the use of two particular operators, the problem of completeness is resolved automatically.

2.4. List of dual sets of q-orthogonal polynomials. In this subsection, we give a list of dual sets of q-polynomials, orthogonal on countable sets of points. Each of these dual pairs will be considered in detail in the subsequent sections. In particular, orthogonality relations for them will be explicitly derived.

q-polynomials (determinate moment problem)	their duals (indeterminate moment problem)
little q-Jacobi	dual little q-Jacobi
big q-Jacobi	dual big q-Jacobi (two sets)
discrete q-ultraspherical	dual discrete q-ultraspherical
big q-Laguerre	q-Meixner (two sets)
alternative q-Charlier	dual alternative q-Charlier
Al-Salam–Carlitz I	q-Charlier (two sets)
little q-Laguerre	Al-Salam–Carlitz II

Let us exhibit these dual pairs explicitly.

Little q-Jacobi polynomials and their duals. Little q-Jacobi polynomials, given by the formula

$$p_n(\lambda; a, b|q) := {}_2\phi_1(q^{-n}, abq^{n+1};\ aq;\ q, q\lambda), \tag{2.10}$$

are orthogonal for $0 < a < q^{-1}$ and $b < q^{-1}$. The dual little q-Jacobi polynomials, corresponding to the polynomials (2.10) with the same values of the parameters a and b, are given as

$$d_n(\mu(m); a, b|q) := {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n};\ bq;\ q, q^n/a),$$

where $\mu(m) = q^{-m} + abq^{m+1}$. Since these polynomials are absent in the Askey q-scheme [27], we give the orthogonality relation for these polynomials:

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(abq, bq; q)_m}{(1-abq)(aq, q; q)_m}\, a^m\, q^{m^2}\, d_n(\mu(m))\, d_{n'}(\mu(m))$$

$$= \frac{(abq^2; q)_\infty}{(aq; q)_\infty} \frac{(q; q)_n (aq)^{-n}}{(b; q)_n}\, \delta_{nn'}.$$

The polynomials $d_n(\mu(m))$ correspond to the indeterminate moment problem and the orthogonality measure here is extremal. The duality of these polynomials to the set of the little q-Jacobi polynomials was first observed in [6] (see also [8]).

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Big q-Jacobi polynomials and their duals. Big q-Jacobi polynomials, given by the formula

$$P_n(\lambda; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, \lambda;\ aq, cq;\ q, q), \tag{2.11}$$

are orthogonal for $0 < a, b < q^{-1}$ and $c > 0$. The dual big q-Jacobi polynomials, associated with the polynomials (2.11) with the same values of the parameters a, b, c, are given as

$$D_n(\mu(m); a, b, c|q) := {}_3\phi_2(q^{-m}, abq^{m+1}, q^{-n};\ aq, abq/c;\ q, aq^{n+1}/c), \tag{2.12}$$

where $\mu(m) = q^{-m} + abq^{m+1}$. The second set of dual polynomials with respect to (2.11) is obtained from the polynomials (2.12) by the replacements $a, b, c \to b, a, ab/c$, respectively. Again, since these polynomials are absent in the q-Askey scheme, we give here the orthogonality relation for the polynomials (2.12):

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(aq, abq, abq/c; q)_m}{(1-abq)(bq, cq, q; q)_m} (-c/a)^m\, q^{m(m-1)/2}\, D_n(\mu(m))\, D_{n'}(\mu(m))$$

$$= \frac{(abq^2, c/a; q)_\infty}{(bq, cq; q)_\infty} \frac{(aq/c, q; q)_n}{(aq, abq/c; q)_n q^n} \delta_{nn'}.$$

The polynomials $D_n(\mu(m))$ correspond to the indeterminate moment problem and the orthogonality measure here is not extremal.

Discrete q-ultraspherical polynomials and their duals. Discrete q-ultraspherical polynomials $C_n^{(a)}(x; q)$, $a > 0$, are a particular case of the big q-Jacobi polynomials

$$C_n^{(a^2)}(x; q) = P_n(x : a, a, -a; q) = {}_3\phi_2\,(q^{-n}, a^2q^{n+1}, x;\ aq, -aq;\ q, q).$$

An orthogonality relation for $C_n^{(a)}(x; q)$ follows from that for the big q-Jacobi polynomials and it holds for positive values of a. We can consider the polynomials $C_n^{(a)}(x; q)$ also for other values of a. In particular, they are orthogonal for imaginary values of a and x. In order to dispense with imaginary numbers in this case, the following notation is introduced:

$$\tilde{C}_n^{(a^2)}(x; q) := (-\mathrm{i})^n C_n^{(-a^2)}(\mathrm{i}x; q) = (-\mathrm{i})^n P_n(\mathrm{i}x; \mathrm{i}a, \mathrm{i}a, -\mathrm{i}a; q), \tag{2.13}$$

The orthogonality relation for them is of the form

$$\sum_{s=0}^{\infty} \frac{(-aq^2; q^2)_s\, q^s}{(q^2; q^2)_s} \tilde{C}_{2k+1}^{(a)}(\sqrt{a}\, q^{s+1}; q) \tilde{C}_{2k'+1}^{(a)}(\sqrt{a}\, q^{s+1}; q)$$

$$= \frac{(-aq^3; q^2)_\infty}{(q; q^2)_\infty} \frac{(1+aq)\, a^{2k+1}}{(1+aq^{4k+3})} \frac{(q; q)_{2k+1}}{(-aq; q)_{2k+1}}\, q^{(k+2)(2k+1)} \delta_{kk'}.$$

The formula

$$D_n^{(a^2)}(\mu(x; a^2)|q) := D_n(\mu(x; a^2); a, a, -a|q) := {}_3\phi_2 \left(\begin{matrix} q^{-x}, a^2q^{x+1}, q^{-n} \\ aq, -aq \end{matrix} \middle|\, q, -q^{n+1} \right),$$

where $\mu(x; a^2) = q^{-x} + a^2q^{x+1}$ and $D_n(\mu(x; a^2))$ are dual big q-Jacobi polynomials, gives dual discrete q-ultraspherical polynomials. They correspond to indeterminate moment problem. The dual orthogonality relation for them (when $a^2 > 0$) follows from the orthogonality relation for dual big q-Jacobi polynomials.

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For the polynomials $D_n^{(a^2)}(\mu(x;a^2)|q)$ with imaginary a we have

$$\tilde{D}_n^{(a^2)}(\mu(x;-a^2)|q) := D_n(\mu(x;-a^2);\mathrm{i}a,\mathrm{i}a,-\mathrm{i}a|q)$$

$$:= {}_3\phi_2\left(\begin{matrix} q^{-x}, -a^2q^{x+1}, q^{-n} \\ \mathrm{i}aq, -\mathrm{i}aq \end{matrix}\,\middle|\, q, -q^{n+1}\right).$$

These polynomials are dual to the polynomials $\tilde{C}_n^{(a^2)}(x;q)$ from (2.13). In this case there are also infinitely many orthogonality relations, which are considered in section 5.

Big q-Laguerre polynomials and q-Meixner polynomials. Big q-Laguerre polynomials, given by the formula

$$P_n(\lambda;a,b;q) := {}_3\phi_2(q^{-n},0,\lambda;\ aq,bq;\ q,q),$$

are orthogonal for $0 < a < q^{-1}$ and $b < 0$. The dual polynomials coincide with q-Meixner polynomials $M_n(q^{-x};a,-b/a;q)$ and $M_n(q^{-x};b,-a/b;q)$, where

$$M_n(q^{-x};a,b;q) := {}_2\phi_1(q^{-n},q^{-x};\ aq;\ q,-q^{n+1}/b).$$

We obtain orthogonality relations for the q-Meixner polynomials $M_n(q^{-x};a,b;q)$ with $b < 0$ and $b > 0$ in section 6.

The duality relation between big q-Laguerre polynomials and q-Meixner polynomials was studied in [2]. The appearance of q-Meixner polynomials as a dual family with respect to the big q-Laguerre polynomials is quite natural because the transformation $q \to q^{-1}$ interrelates these two sets of polynomials, that is,

$$M_n(x;b,c;q^{-1}) = (q^{-n}/b;q)_n\, P_n(qx/b;1/b,-c;q).$$

Alternative q-Charlier polynomials and their duals. Alternative q-Charlier polynomials are given by the formula

$$K_n(\lambda;a;q) := {}_2\phi_1(q^{-n},-aq^n;\ 0;\ q,q\lambda)\,.$$

They are orthogonal for $a > 0$. Their duals are the polynomials

$$d_n(\mu(m);a;q) := {}_3\phi_0(q^{-m},-a\,q^m,q^{-n};\ -\,;\ q,-q^n/a)\,, \qquad \mu(m) := q^{-m} - a\,q^m\,,$$

which correspond to the indeterminate moment problem (see [10]). They are also absent in the q-Askey scheme. The orthogonality relation for these polynomials is

$$\sum_{m=0}^{\infty} \frac{(1+aq^{2m})a^m}{(-aq^m;q)_\infty (q;q)_m}\, q^{m(3m-1)/2} d_n(\mu(m)) d_{n'}(\mu(m)) = \frac{(q;q)_n}{a^n q^{n(n+1)/2}}\,\delta_{nn'}, \quad a > 0.$$

Al-Salam–Carlitz I polynomials and q-Charlier polynomials. Al-Salam–Carlitz I polynomials, given by the formula

$$U_n^{(a)}(x;\,q) := (-a)^n q^{n(n-1)/2}\, {}_2\phi_1(q^{-n},x^{-1};\ 0;\ q;\ xq/a),$$

are orthogonal for $a < 0$. There are two sets of dual polynomials [25]. They coincide with two sets of q-Charlier polynomials $C_n(q^{-x};\,-a;\,q)$ and $C_n(q^{-x};\,-1/a;\,q)$, where

$$C_n(q^{-x};\,a;\,q) := {}_2\phi_1(q^{-n},q^{-x};\ 0;\ q;-q^{n+1}/a).$$

Little q-Laguerre polynomials and Al-Salam–Carlitz II polynomials. The little q-Laguerre polynomials are given by the formula

$$p_n(x;\, a|q) := {}_2\phi_1(q^{-n}, 0;\; aq;\; q; qx) = (a^{-1}q^{-n}; q)_n^{-1}\; {}_2\phi_0(q^{-n}, x^{-1};\; -;\; q; x/a).$$

They are orthogonal for $0 < a < q^{-1}$. The dual polynomials with respect to them are the Al-Salam–Carlitz II polynomials (see [7])

$$V_n^{(a)}(x;\, q) := (-a)^n q^{-n(n-1)/2}\, {}_2\phi_0(q^{-n}, x;\; -;\; q; q^n/a).$$

3. Little q-Jacobi polynomials and their duals.

3.1. Pair of operators (I_1, J). Let $\mathcal{H}$ be a separable complex Hilbert space with an orthonormal basis f_n, $n = 0, 1, 2, \cdots$. The basis determines uniquely a scalar product in $\mathcal{H}$. In order to deal with a Hilbert space of functions on a real line, we fix a real number a such that $0 < a < q^{-1}$ and realize our Hilbert space in such a way that basis elements f_n are monomials:

$$f_n \equiv f_n(x) := c_n\, x^n,$$

where

$$c_n = a^{-n/4}\,\frac{(aq; q)_n^{1/2}}{(q; q)_n^{1/2}}\,, \quad n = 0, 1, 2, 3, \cdots .$$

Thus, in fact, our Hilbert space depends on the number a and can be denoted as $\mathcal{H}_a$.

We fix two real parameters a and b such that $b < q^{-1}$, $0 < a < q^{-1}$, and define on $\mathcal{H} \equiv \mathcal{H}_a$ two operators. The first one, denoted as q^{J_0} and taken from the theory of representations of quantum group $U_q(\mathrm{su}_{1,1})$, acts on the basis elements as

$$q^{J_0} f_n = (qa)^{1/2} q^n f_n. \tag{3.1}$$

The second operator, denoted as I_1, is given by the formula

$$I_1\, f_n = -a_n f_{n+1} - a_{n-1} f_{n-1} + b_n f_n, \tag{3.2}$$

where

$$a_n = a^{1/2} q^{n+1/2}\,\frac{\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})(1-abq^{n+1})}}{(1-abq^{2n+2})\sqrt{(1-abq^{2n+1})(1-abq^{2n+3})}}\,,$$

$$b_n = \frac{q^n}{1-abq^{2n+1}}\left(\frac{(1-aq^{n+1})(1-abq^{n+1})}{1-abq^{2n+2}} + a\frac{(1-q^n)(1-bq^n)}{1-abq^{2n}}\right).$$

The expressions for a_n and b_n are well defined. The operator I_1 is symmetric.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator I_1 is bounded. Therefore, we assume that it is defined on the whole space $\mathcal{H}$. For this reason, I_1 is a self-adjoint operator. Let us show that I_1 is a Hilbert–Schmidt operator (we remind that a bounded self-adjoint operator is a Hilbert–Schmidt operator if a sum of its squared matrix elements in an orthonormal basis is finite; the spectrum of such an operator is discrete, with a single accumulation point at 0). For the coefficients a_n and b_n from (3.2), we have

$$a_{n+1}/a_n \to q, \;\; b_{n+1}/b_n \to q \quad \text{when} \quad n \to \infty.$$

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Therefore, for the sum of all matrix elements of the operator I_1 in the canonical basis we have $\sum_n (2a_n + b_n) < \infty$. This means that I_1 is a Hilbert–Schmidt operator. Thus, the spectrum of I_1 is discrete and has a single accumulation point at 0. Moreover, a spectrum of I_1 is simple, since I_1 is representable by a Jacobi matrix with $a_n \neq 0$ (see [15], Chapter VII).

To find eigenfunctions $\xi_\lambda(x)$ of the operator I_1, $I_1\xi_\lambda(x) = \lambda\xi_\lambda(x)$, we set

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} \beta_n(\lambda) f_n(x).$$

Acting by the operator I_1 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda)\,(a_n f_{n+1} + a_{n-1} f_{n-1} - b_n f_n) = -\lambda \sum_{n=0}^{\infty} \beta_n(\lambda) f_n,$$

where a_n and b_n are the same as in (3.2). Collecting in this identity all factors, which multiply f_n with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)a_n + \beta_{n-1}(\lambda)a_{n-1} - \beta_n(\lambda)b_n = -\lambda\beta_n(\lambda).$$

The substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n\,(1-abq^{2n+1})}{(bq, q; q)_n\,(1-abq)(aq)^n} \right)^{1/2} \beta'_n(\lambda)$$

reduces this relation to the following one

$$A_n\beta'_{n+1}(\lambda) + C_n\beta'_{n-1}(\lambda) - (A_n + C_n)\beta'_n(\lambda) = -\lambda\beta'_n(\lambda)$$

with

$$A_n = \frac{q^n(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = \frac{aq^n(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

This is the recurrence relation for the little q-Jacobi polynomials

$$p_n(\lambda; a, b|q) := {}_2\phi_1(q^{-n}, abq^{n+1};\ aq;\ q, q\lambda) \tag{3.3}$$

(see, for example, formula (7.3.1) in [21]). Therefore, $\beta'_n(\lambda) = p_n(\lambda; a, b|q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n\,(1-abq^{2n+1})}{(bq, q; q)_n\,(1-abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q). \tag{3.4}$$

For the eigenfunctions $\xi_\lambda(x)$ we have the expression

$$\begin{aligned}\xi_\lambda(x) &= \sum_{n=0}^{\infty} \left(\frac{(abq, aq; q)_n\,(1-abq^{2n+1})}{(bq, q; q)_n\,(1-abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q)\, f_n(x) \\ &= \sum_{n=0}^{\infty} a^{-n/4}\, \frac{(aq;q)_n}{(q;q)_n} \left(\frac{(abq; q)_n\,(1-abq^{2n+1})}{(bq; q)_n\,(1-abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q) x^n.\end{aligned} \tag{3.5}$$

Since the spectrum of the operator I_1 is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}$. This discrete set of functions determines a spectrum of I_1.

Now we look for a spectrum of the operator I_1 and for a set of polynomials, dual to the little q-Jacobi polynomials. To this end we use the action of the operator

$$J := (qa)^{1/2} q^{-J_0} + (qa)^{-1/2} ab\, q^{J_0+1}$$

upon the eigenfunctions $\xi_\lambda(x)$, which belong to the Hilbert space $\mathcal{H}$. In order to find how this operator acts upon these functions, one can use the q-difference equation

$$(q^{-n} + abq^{n+1})\, p_n(\lambda) = a\lambda^{-1}(bq\lambda - 1)\, p_n(q\lambda) + \lambda^{-1}(1+a)\, p_n(\lambda) + \lambda^{-1}(\lambda - 1)\, p_n(q^{-1}\lambda) \tag{3.6}$$

for the little q-Jacobi polynomials $p_n(\lambda) \equiv p_n(\lambda; a, b|q)$ (see, for example, formula (3.12.5) in [27]). Multiply both sides of (3.6) by $d_n\, f_n(x)$ and sum up over n, where d_n are the coefficients of $p_n(\lambda; a, b|q)$ in the expression (3.4) for the $\beta_n(\lambda)$. Taking into account the first line in formula (3.5) and the fact that $J f_n(x) = (q^{-n} + abq^{n+1}) f_n(x)$, one obtains the relation

$$J\,\xi_\lambda(x) = a\lambda^{-1}(bq\lambda - 1)\,\xi_{q\lambda}(x) + \lambda^{-1}(1+a)\,\xi_\lambda(x) + \lambda^{-1}(\lambda - 1)\,\xi_{q^{-1}\lambda}(x). \tag{3.7}$$

It will be shown in the next section that the spectrum of the operator I_1 consists of the points $\lambda = q^n$, $n = 0, 1, 2, \cdots$. Thus, we see that the pair of the operators I_1 and J form a Leonard pair (see [35], where P. Terwilliger has actually introduced this notion in an effort to interpret the results of D. Leonard [28]; see also [37], which contains a review on how one can employ Leonard pairs to describe properties of orthogonal polynomials). We remind to the reader that a pair of operators R_1 and R_2, acting on a linear space $\mathcal{L}$, is a Leonard pair if

(a) there exists a basis in $\mathcal{L}$, with respect to which the operator R_1 is diagonal, and the operator R_2 has the form of a Jacobi matrix;

(b) there exists another basis of $\mathcal{L}$, with respect to which the operator R_2 is diagonal, and the operator R_1 has the form of a Jacobi matrix.

Properties of Leonard pairs of operators in finite dimensional spaces are studied in detail. Leonard pairs in infinite dimensional spaces are more complicated and only some isolated results are known in this case (see, for example, [36]).

3.2. Spectrum of I_1 and orthogonality of little q-Jacobi polynomials. The aim of this section is to find, by using the Leonard pair (I_1, J), a basis in the Hilbert space $\mathcal{H}$, which consists of eigenfunctions of the operator I_1 in a normalized form, and to derive explicitly the unitary matrix U, connecting this basis with the canonical basis f_n, $n = 0, 1, 2, \cdots$, in $\mathcal{H}$. This matrix directly leads the orthogonality relation for the little q-Jacobi polynomials.

Let us analyze a form of spectrum of the operator I_1. If λ is a spectral point of the operator I_1, then (as it is easy to see from (3.7)) a successive action by the operator J upon the function (eigenfunction of I_1) ξ_λ leads to the functions

$$\xi_{q^m\lambda}, \quad m = 0, \pm 1, \pm 2, \cdots,$$

which are eigenfunctions of I_1 with eigenvalues $q^m\lambda$. However, since I_1 is a trace class operator, not all these points can belong to the spectrum of I_1, since $q^{-m}\lambda \to \infty$ when $m \to \infty$ if $\lambda \neq 0$. This means that under a successive action by I_1 upon ξ_λ, on some step the last term in (3.7) must vanish. Thus, under the action by I_1 upon $\xi_{\lambda'}$ for some λ' the coefficient $\lambda' - 1$ of $\xi_{q^{-1}\lambda'}(x)$ in (3.7) vanishes. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for the coefficient of $\xi_{q^{-1}\lambda'}(x)$ in (3.7) to vanish, that is, the point

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$\lambda = 1$ is a spectral point for the operator I_1. Let us show that the corresponding eigenfunction $\xi_1(x) \equiv \xi_{q^0}(x)$ belongs to the Hilbert space $\mathcal{H}$.

Observe that by formula (II.6) of Appendix II in [21], one has

$$p_n(1; a, b|q) = {}_2\phi_1(q^{-n}, abq^{n+1};\ aq;\ q, q) = \frac{(b^{-1}q^{-n}; q)_n}{(aq; q)_n}(abq^{n+1})^n.$$

Since $(b^{-1}q^{-n}; q)_n = (bq; q)_n(-b^{-1}q^{-1})^n q^{-n(n-1)/2}$, this means that

$$p_n(1; a, b|q) = \frac{(bq; q)_n}{(aq; q)_n}(-a)^n q^{n(n+1)/2}.$$

Therefore, due to (3.5) for the scalar product $\langle \xi_1(x), \xi_1(x)\rangle$ we have

$$\langle \xi_1(x), \xi_1(x)\rangle = \sum_{n=0}^{\infty} \frac{(abq, aq; q)_n\,(1 - abq^{2n+1})}{(bq, q; q)_n\,(1 - abq)(aq)^n}\, p_n^2(1; a, b|q)$$

$$= \sum_{n=0}^{\infty} \frac{(abq, bq; q)_n\,(1 - abq^{2n+1})}{(aq, q; q)_n(1 - abq)}\, a^n q^{n^2} = \frac{(abq^2; q)_\infty}{(aq; q)_\infty}. \tag{3.8}$$

The last leg of this equality is obtained from formula (9.17) of Appendix. Thus, the series (3.9) converges and, therefore, the point $\lambda = 1$ actually belongs to the spectrum of the operator I_1.

Let us find other spectral points of the operator I_1 (recall that a spectrum of I_1 is discrete). Setting $\lambda = 1$ in (3.7), we see that the operator J transforms $\xi_{q^0}(x)$ into a linear combination of the functions $\xi_q(x)$ and $\xi_{q^0}(x)$. Moreover, $\xi_q(x)$ belongs to the Hilbert space $\mathcal{H}$, since the series

$$\langle \xi_q, \xi_q\rangle = \sum_{n=0}^{\infty} \frac{(abq, aq; q)_n\,(1 - abq^{2n+1})}{(bq, q; q)_n\,(1 - abq)(aq)^n} p_n^2(q; a, b|q) < \infty$$

is majorized by the corresponding series for $\xi_{q^0}(x)$, considered above. Therefore, $\xi_q(x)$ belongs to the Hilbert space $\mathcal{H}$ and the point q is an eigenvalue of the operator I_1. Similarly, setting $\lambda = q$ in (3.7), we find that $\xi_{q^2}(x)$ is an eigenfunction of I_1 and the point q^2 belongs to the spectrum of I_1. Repeating this procedure, we find that $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, are eigenfunctions of I_1 and the set q^n, $n = 0, 1, 2, \cdots$, belongs to the spectrum of I_1. So far, we do not know yet whether other spectral points exist or not.

The functions $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, are linearly independent elements of the space $\mathcal{H}$ (since they correspond to different eigenvalues of the self-adjoint operator I_1). Suppose that values q^n, $n = 0, 1, 2, \cdots$, constitute a whole spectrum of the operator I_1. Then the set of functions $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, is a basis in the Hilbert space $\mathcal{H}$. Introducing the notation $\Xi_n := \xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, we find from (3.7) that

$$J\,\Xi_n = -aq^{-n}(1 - bq^{n+1})\,\Xi_{n+1} + q^{-n}(a+1)\,\Xi_n - q^{-n}(1 - q^n)\,\Xi_{n-1}.$$

As we see, the matrix of the operator J in the basis Ξ_n, $n = 0, 1, 2, \cdots$, is not symmetric, although in the initial basis f_n, $n = 0, 1, 2, \cdots$, it was symmetric. The reason is that the matrix $A \equiv (a_{mn})$ with entries

$$a_{mn} := \beta_m(q^n), \quad m, n = 0, 1, 2, \cdots,$$

where $\beta_m(q^n)$ are the coefficients (3.4) in the expansion $\xi_{q^n}(x) = \sum_m \beta_m(q^n) f_m(x)$, is not unitary. (This matrix connects the bases $\{f_n\}$ and $\{\Xi_n\}$.) It is equivalent to the statement that the basis $\Xi_n := \xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, is not normalized. To normalize it, one has to multiply Ξ_n by corresponding numbers c_n (which are not known at this moment). Let $\hat{\Xi}_n = c_n \Xi_n$, $n = 0, 1, 2, \cdots$, be a normalized basis. Then the matrix of the operator J is symmetric in this basis. Since J has in the basis $\{\hat{\Xi}_n\}$ the form

$$J\,\hat{\Xi}_n = -c_{n+1}^{-1} c_n a q^{-n}(1 - bq^{n+1})\,\hat{\Xi}_{n+1} + q^{-n}(a+1)\,\hat{\Xi}_n - c_{n-1}^{-1} c_n q^{-n}(1 - q^n)\,\hat{\Xi}_{n-1},$$

then its symmetricity means that

$$c_{n+1}^{-1} c_n a q^{-n}(1 - bq^{n+1}) = c_n^{-1} c_{n+1} q^{-n-1}(1 - q^{n+1}),$$

that is, $c_n/c_{n-1} = \sqrt{aq(1 - bq^n)/(1 - q^n)}$. Therefore,

$$c_n = c(aq)^{n/2} \frac{(bq;q)_n^{1/2}}{(q;q)_n^{1/2}},$$

where c is a constant.

Now instead of the expansion (3.5) we have the expansions

$$\hat{\xi}_{q^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\, \beta_m(q^n) f_m(x), \tag{3.9}$$

which connect two orthonormal bases in the space $\mathcal{H}$. This means that the matrix $(\hat{a}_{mn})$, $m, n = 0, 1, 2, \cdots$, with entries

$$\hat{a}_{mn} = c_n \beta_m(q^n) \tag{3.10}$$

$$= c\left((aq)^{n-m} \frac{(bq;q)_n}{(q;q)_n} \frac{(abq, aq;q)_m\,(1 - abq^{2m+1})}{(bq, q;q)_m\,(1 - abq)} \right)^{1/2} p_m(q^n; a, b|q),$$

is unitary, provided that the constant c is appropriately chosen. In order to calculate this constant, we use the relation $\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1$ for $n = 0$. Then the sum in this relation is a multiple of the sum in (3.8) and, consequently,

$$c = \frac{(aq;q)_\infty^{1/2}}{(abq;q)_\infty^{1/2}}.$$

Thus the c_n in (3.9) and (3.10) are real and equal to

$$c_n = \left(\frac{(aq;q)_\infty}{(abq;q)_\infty} \frac{(bq;q)_n (aq)^n}{(q;q)_n} \right)^{1/2}.$$

The matrix $(\hat{a}_{mn})$ with entries (3.10) is orthogonal, that is,

$$\sum_n \hat{a}_{mn} \hat{a}_{m'n} = \delta_{mm'}, \quad \sum_m \hat{a}_{mn} \hat{a}_{mn'} = \delta_{nn'}. \tag{3.11}$$

Substituting into the first sum over n in (3.11) the expressions for $\hat{a}_{mn}$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n (aq)^n}{(q;q)_n}\, p_m(q^n; a, b|q)\, p_{m'}(q^n; a, b|q) \tag{3.12}$$

$$= \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(1-abq)(aq)^m\,(bq,q;q)_m}{(1-abq^{2m+1})\,(abq,aq;q)_m}\,\delta_{mm'}\,,$$

which must yield the orthogonality relation for the little q-Jacobi polynomials. An only gap, which appears here, is the following. We have assumed that the points q^n, $n=0,1,2,\cdots$, exhaust the whole spectrum of the operator I_1. Let us show that this is the case.

Recall that the self-adjoint operator I_1 is represented by a Jacobi matrix in the basis f_n, $n=0,1,2,\cdots$. According to the theory of operators of such type (see, for example, [15], Chapter VII; a short explanation is given in section 2), eigenfunctions ξ_λ of I_1 are expanded into series in the monomials f_n, $n=0,1,2,\cdots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some positive measure $d\mu(\lambda)$ (moreover, for self-adjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple. Let us apply these assertions to the operator I_1.

We have found that the spectrum of I_1 contains the points q^n, $n=0,1,2,\cdots$. If the operator I_1 had other spectral points x, then on the left-hand side of (3.15) there would be other summands $\mu_{x_k}\,p_m(x_k;a,b|q)\,p_{m'}(x_k;a,b|q)$, corresponding to these additional points. Let us show that these additional summands do not appear. To this end we set $m=m'=0$ in the relation (3.12) with the additional summands. Since $p_0(x;a,b|q)=1$, we have the equality

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n (aq)^n}{(q;q)_n} + \sum_k \mu_{x_k} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty}.$$

According to the q-binomial theorem (see formula (1.3.2) in [21]), we have

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n\,(aq)^n}{(q;q)_n} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty}. \tag{3.13}$$

Hence, $\sum_k \mu_{x_k}=0$ and this means that additional summands do not appear in (3.12) and it does represent the orthogonality relation for the little q-Jacobi polynomials.

By using the operators I_1 and J, which form a Leonard pair of infinite dimensional symmetric operators, we thus derived the orthogonality relation for little q-Jacobi polynomials.

The orthogonality relation for the little q-Jacobi polynomials is given by formula (3.12). Due to this orthogonality, we arrive at the following statement: *The spectrum of the operator I_1 coincides with the set of points q^n, $n=0,1,2,\cdots$. The spectrum is simple and has one accumulation point at 0.*

3.3. Dual little q-Jacobi polynomials. Now we consider the second identity in (3.11), which gives the orthogonality relation for the matrix elements $\hat{a}_{mn}$, considered as functions of m. Up to multiplicative factors these functions coincide with

$$F_n(x;a,b|q) := {}_2\phi_1(x, abq/x;\; aq;\; q, q^{n+1}), \tag{3.14}$$

considered on the set $x\in\{q^{-m}\,|\,m=0,1,2,\cdots\}$. Consequently,

$$\hat{a}_{mn} = \left(\frac{(aq;q)_\infty}{(abq;q)_\infty} \frac{(bq;q)_n}{(q;q)_n} (aq)^{n-m} \frac{(abq,aq;q)_m\,(1-abq^{2m+1})}{(bq,q;q)_m} \right)^{1/2} F_n(q^{-m};a,b|q)$$

and the second identity in (3.11) gives the orthogonality relation for the functions (3.14):

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})\,(abq,aq;q)_m}{(1-abq)(aq)^m\,(bq,q;q)_m} F_n(q^{-m};a,b|q)\,F_{n'}(q^{-m};a,b|q) \tag{3.15}$$

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$$= \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(q;q)_n (aq)^{-n}}{(bq;q)_n} \, \delta_{nn'} .$$

The functions $F_n(x;a,b|q)$ can be represented in another form. Indeed, one can use the relation (III.8) of Appendix III in [21] in order to obtain that

$$F_n(q^{-m};a,b|q) = \frac{(b^{-1}q^{-m};q)_m}{(aq;q)_m} (abq^{m+1})^m \, {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n};\; bq;\; q, q^n/a)$$

$$= \frac{(-1)^m \, (bq;q)_m}{(aq;q)_m} \, a^m \, q^{m(m+1)/2} {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n};\; bq;\; q, q^n/a). \tag{3.16}$$

The basic hypergeometric function ${}_3\phi_1$ in (3.16) is a polynomial of degree n in the variable $\mu(m) := q^{-m} + ab\, q^{m+1}$, which represents a q-quadratic lattice; we denote it as

$$d_n(\mu(m);a,b|q) := {}_3\phi_1(q^{-m}, ab\, q^{m+1}, q^{-n};\; bq;\; q, q^n/a) \, . \tag{3.17}$$

Then formula (3.15) yields the orthogonality relation

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(abq, bq;q)_m}{(1-abq)(aq,q;q)_m} \, a^m \, q^{m^2} \, d_n(\mu(m)) \, d_{n'}(\mu(m))$$

$$= \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \, \frac{(q;q)_n (aq)^{-n}}{(bq;q)_n} \, \delta_{nn'} \tag{3.18}$$

for the polynomials (3.17). We call the polynomials $d_n(\mu(m);a,b|q)$ *dual little q-Jacobi polynomials*.

Note that these polynomials can be expressed in terms of the Al-Salam–Chihara polynomials

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n} \, {}_3\phi_2 \left(\left. \begin{matrix} q^{-n}, az, az^{-1} \\ ab, 0 \end{matrix} \right| q, q \right), \quad x = \frac{1}{2}(z+z^{-1}),$$

with the parameter $q > 1$. An explicit relation between them is

$$d_n(\mu(x); \beta/\alpha, \, 1/\alpha\beta q \,|\, q) = \frac{q^{n(n-1)/2}}{(-\beta)^n (1/\alpha\beta;q)_n} \, Q_n(\alpha\mu(x)/2; \alpha, \beta|q^{-1}).$$

Ch. Berg and M. E. H. Ismail studied this type of Al-Salam–Chihara polynomials in [17] and derived complex orthogonality measures for them. But [17] does not contain any discussion of the duality of this family of polynomials with respect to little q-Jacobi polynomials.

Observe that the dual polynomials (3.17) can be also expressed in terms of the little q-Jacobi polynomials (3.3):

$$d_n(\mu(m);a,b|q) = \frac{(aq;q)_m}{(bq/c;q)_m} \, (-a)^{-m} \, q^{-m(m+1)/2} \, p_m(q^n;a,b|q).$$

A recurrence relation for the polynomials $d_n(\mu(m);a,b|q)$ is derived from formula (3.6). It has the form

$$(q^{-m} + abq^{m+1}) \, d_n(\mu(m)) = -\, a\, q^{-n}(1-bq^{n+1}) \, d_{n+1}(\mu(m))$$

$$+ q^{-n}(1+a)\, d_n(\mu(m)) - q^{-n}(1-q^n)\, d_{n-1}(\mu(m)),$$

where $d_n(\mu(m)) \equiv d_n(\mu(m); a, b|q)$. Comparing this relation with the recurrence relation (3.69) in [5], we see that the polynomials (3.17) are multiple to the polynomials (3.67) in [5]. Moreover, if one takes into account this multiplicative factor, the orthogonality relation (3.18) for polynomials (3.17) turns into relation (3.82) for the polynomials (3.67) in [5], although the derivation of the orthogonality relation in [5] is more complicated than our derivation of (3.18). The authors of [5] do not give an explicit form of their polynomials in the form similar to (3.17). Concerning the polynomials (3.67) in [5] see also [22].

Let $\mathfrak{l}^2$ be the Hilbert space of functions on the set $m = 0, 1, 2, \cdots$ with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})\,(abq, bq; q)_m}{(1-abq)\,(aq, q; q)_m}\, a^m\, q^{m^2}\, f_1(m)\, \overline{f_2(m)},$$

where weight function is taken from (3.18). The polynomials (3.17) are in one-to-one correspondence with the columns of the unitary matrix $(\hat{a}_{mn})$ and the orthogonality relation (3.18) is equivalent to the orthogonality of these columns. Due to (3.11) the columns of the matrix $(\hat{a}_{mn})$ form an orthonormal basis in the Hilbert space of sequences $\mathbf{a} = \{a_n \,|\, n = 0, 1, 2, \cdots\}$ with the scalar product $\langle \mathbf{a}, \mathbf{a}' \rangle = \sum_n a_n\, a'_n$. This assertion is equivalent to the following one: the set of polynomials $d_n(\mu(m); a, b|q)$, $n = 0, 1, 2, \cdots$, form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$. This means that *the point measure in* (3.18) *is extremal for the dual little q-Jacobi polynomials* $d_n(\mu(m); a, b|q)$.

4. Big q-Jacobi polynomials and their duals.

4.1. Pair of operators (I_2, J). We fix three real numbers a, b and c such that $0 < a < q^{-1}$, $0 < b < q^{-1}$, $c < 0$ and consider on the Hilbert space $\mathcal{H} \equiv \mathcal{H}_a$, introduced in subsection 3.1, the following symmetric operator I_2:

$$I_2\, f_n = a_n f_{n+1} + a_{n-1} f_{n-1} - b_n f_n, \tag{4.1}$$

where

$$a_{n-1} = (-acq^{n+1})^{1/2}\, \frac{\sqrt{(1-q^n)(1-aq^n)(1-bq^n)(1-abq^n)(1-cq^n)(1-abc^{-1}q^n)}}{(1-abq^{2n})\,\sqrt{(1-abq^{2n-1})(1-abq^{2n+1})}},$$

$$b_n = \frac{(1-aq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})} - acq^{n+1}\frac{(1-q^n)(1-bq^n)(1-abq^n/c)}{(1-abq^{2n})(1-abq^{2n+1})} - 1\,.$$

This operator is bounded. Therefore, we assume that it is defined on the whole Hilbert space $\mathcal{H}$. This means that I_2 is a self-adjoint operator. Actually, I_2 is a Hilbert–Schmidt operator. To show this we note that for the coefficients a_n and b_n from (4.1) one obtains that

$$a_{n+1}/a_n \to q^{1/2}, \;\; b_{n+1}/b_n \to q \;\;\text{ when }\;\; n \to \infty.$$

Therefore, $\sum_n (2a_n + b_n) < \infty$ and this means that I_2 is a Hilbert–Schmidt operator. Thus, the spectrum of I_2 is simple (since it is representable by a Jacobi matrix with $a_n \neq 0$), discrete and have a single accumulation point at 0.

To find eigenfunctions $\psi_\lambda(x)$ of the operator I_2, $I_2\psi_\lambda(x) = \lambda\psi_\lambda(x)$, we set

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \beta_n(\lambda) f_n(x).$$

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Acting by the operator I_2 on both sides of this relation, one derives that

$$\sum_n \beta_n(\lambda)(a_n f_{n+1} + a_{n-1} f_{n-1} - b_n f_n) = \lambda \sum \beta_n(\lambda) f_n,$$

where a_n and b_n are the same as in (4.1). Collecting in this identity factors, which multiply f_n^l with fixed n, we arrive at the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$a_n \beta_{n+1}(\lambda) + a_{n-1}\beta_{n-1}(\lambda) - b_n \beta_n(\lambda) = \lambda \beta_n(\lambda).$$

Making the substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n \,(1 - abq^{2n+1})}{(abq/c, bq, q; q)_n \,(1-abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4}\, \beta_n'(\lambda)$$

we reduce this relation to the following one

$$A_n \beta_{n+1}'(\lambda) + C_n \beta_{n-1}'(\lambda) - (A_n + C_n - 1) p_n'(\lambda) = \lambda \beta_n'(\lambda)$$

with

$$A_n = \frac{(1-aq^{n+1})(1-cq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})\,(1-abq^{2n+2})},$$

$$C_n = \frac{-acq^{n+1}(1-q^n)(1-bq^n)(1-abc^{-1}q^n)}{(1-abq^{2n})\,(1-abq^{2n+1})}.$$

It is the recurrence relation for the big q-Jacobi polynomials

$$P_n(\lambda; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, \lambda;\; aq, cq;\; q, q) \tag{4.2}$$

introduced by G. E. Andrews and R. Askey [1] (see also formula (7.3.10) in [21]). Therefore, $\beta_n'(\lambda) = P_n(\lambda; a, b, c; q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n \,(1 - abq^{2n+1})}{(abq/c, bq, q; q)_n \,(1-abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4}\, P_n(\lambda; a, b, c; q). \tag{4.3}$$

For the eigenfunctions $\psi_\lambda(x)$ we have the expansion

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \left(\frac{(abq, aq, cq; q)_n \,(1 - abq^{2n+1})}{(abq/c, bq, q; q)_n \,(1-abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4}\, P_n(\lambda; a, b, c; q)\, f_n(x)$$

$$= \sum_{n=0}^{\infty} \frac{(aq;q)_n}{a^{n/4}\,(q;q)_n} \left(\frac{(abq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq; q)_n (1-abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4} P_n(\lambda; a, b, c; q)\, x^n. \tag{4.4}$$

Since the spectrum of the operator I_2 is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}$.

In what follows we intend to study a spectrum of the operator I_2 and to find polynomials, dual to big q-Jacobi polynomials. It can be done with the aid of the operator

$$J := (aq)^{1/2} q^{-J_0} + (aq)^{-1/2} ab\, q^{J_0+1},$$

which has been already used in the previous case in subsection 3.1. In order to determine how this operator acts upon the eigenfunctions $\psi_\lambda(x)$, one can use the q-difference equation

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$$(q^{-n}+abq^{n+1})\,P_n(\lambda)=aq\lambda^{-2}(\lambda-1)(b\lambda-c)\,P_n(q\lambda) \tag{4.5}$$

$$-[\lambda^{-2}acq(1+q)-\lambda^{-1}q(ab+ac+a+c)]\,P_n(\lambda)+\lambda^{-2}(\lambda-aq)(\lambda-cq)\,P_n(q^{-1}\lambda),$$

for the big q-Jacobi polynomials $P_n(\lambda)\equiv P_n(\lambda;a,b,c;q)$ (see, for example, formula (3.5.5) in [27]). Multiply both sides of (4.5) by $k_n\,f_n(x)$, where k_n are the coefficients of $P_n(\lambda;a,b,c;q)$ in the expression (4.3) for the coefficients $\beta_n(\lambda)$, and sum over n. Taking into account formula (4.4) and the fact that $J\,f_n(x)=(q^{-n}+ab\,q^{n+1})\,f_n(x)$, one obtains the relation

$$J\,\psi_\lambda(x)=aq\lambda^{-2}(\lambda-1)(b\lambda-c)\,\psi_{q\lambda}(x) \tag{4.6}$$

$$-[\lambda^{-2}acq(1+q)-\lambda^{-1}q(ab+ac+a+c)]\,\psi_\lambda(x)+\lambda^{-2}(\lambda-aq)(\lambda-cq)\,\psi_{q^{-1}\lambda}(x).$$

It will be shown in the next section that the spectrum of the operator I_2 consists of the points aq^n, cq^n, $n=0,1,2,\cdots$. The matrix of the operator J in the basis of eigenfunctions of I_2 consists of two Jacobi matrices (one corresponds to the spectral points aq^n, $n=0,1,2,\cdots$, and another to the spectral points cq^n, $n=0,1,2,\cdots$). In this case, the operators I_2 and J form some generalization of Leonard pair.

4.2. Spectrum of I_2 and orthogonality of big q-Jacobi polynomials. As in subsection 3.2 one can show that for some value of λ (which must belong to the spectrum) the last term on the right side of (4.6) has to vanish. There are two such values of λ: $\lambda=aq$ and $\lambda=cq$. Let us show that both of these points are spectral points of the operator I_2. Observe that, according to (4.2),

$$P_n(aq;a,b,c;q):={}_2\phi_1(q^{-n},abq^{n+1};\;cq;\;q,q)=\frac{(c/abq^n;q)_n}{(cq;q)_n}(ab)^n\,q^{n(n+1)}.$$

Therefore, since

$$(c/abq^n;q)_n=(abq/c;q)_n(-c/ab)^n q^{-n(n+1)/2}\,,$$

one obtains that

$$P_n(aq;a,b,c;q):=\frac{(abq/c;q)_n}{(cq;q)_n}(-c)^n q^{n(n+1)/2}.$$

Likewise,

$$P_n(cq;a,b,c;q):=\frac{(bq;q)_n}{(aq;q)_n}(-a)^n q^{n(n+1)/2}.$$

Hence, for the scalar product $\langle\psi_{aq}(x),\psi_{aq}(x)\rangle$ we have the expression

$$\sum_{n=0}^{\infty}\frac{(1-abq^{2n+1})\,(abq,aq,cq;q)_n}{(1-abq)(abq/c,bq,q;q)_n\,(-ac)^n}\,q^{-n(n+3)/2}\,P_n^2(aq;a,b,c;q)$$

$$=\sum_{n=0}^{\infty}\frac{(1-abq^{2n+1})\,(abq/c,abq,aq;q)_n}{(1-abq)(bq,cq,q;q)_n\,(-a/c)^n}\,q^{n(n-1)/2}=\frac{(abq^2,c/a;q)_\infty}{(bq,cq;q)_\infty}\,, \tag{4.7}$$

where the relation (9.21) from Appendix has been used. Similarly, for $\langle \psi_{cq}(x), \psi_{cq}(x)\rangle$ one has the expression

$$\sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq,aq,cq;q)_n}{(1-abq)(-ac)^n\,(abq/c,bq,q;q)_n}\, q^{-n(n+3)/2}\, P_n^2(cq;a,b,c;q)$$

$$= \frac{(abq^2,a/c;q)_\infty}{(aq,abq/c;q)_\infty}\,, \tag{4.8}$$

where formula (9.22) from Appendix has been used. Thus, the values $\lambda = aq$ and $\lambda = cq$ are spectral points of the operator I_2.

Let us find other spectral points of I_2. Setting $\lambda = aq$ in (4.6), we see that the operator J transforms $\psi_{aq}(x)$ into a linear combination of the functions $\psi_{aq^2}(x)$ and $\psi_{aq}(x)$. We have to show that $\psi_{aq^2}(x)$ also belongs to the Hilbert space $\mathcal{H}$, that is, that

$$\langle \psi_{aq^2}, \psi_{aq^2}\rangle = \sum_{n=0}^{\infty} \frac{(abq,aq,cq;q)_n(1-abq^{2n+1})}{(abq/c,bq,q;q)_n(1-abq)(-ac)^n}\, q^{-n(n+3)/2}\, P_n^2(aq^2;a,b,c;q) < \infty.$$

In order to achieve this we note that since $(aq^2;q)_k = (aq;q)_k(1-aq^{k+1})/(1-aq)$, we have

$$P_n(aq^2;a,b,c;q) = \sum_{k=0}^{n} \frac{1-aq^{k+1}}{1-aq}\, \frac{(q^{-n};q)_k(abq^{n+1};q)_k(aq;q)_k}{(aq;q)_k(cq;q)_k}\, \frac{q^k}{(q;q)_k}$$

$$\le \frac{1}{1-aq}\sum_{k=0}^{n} \frac{(q^{-n};q)_k(abq^{n+1};q)_k(aq;q)_k}{(aq;q)_k(cq;q)_k}\, \frac{q^k}{(q;q)_k} = (1-aq)^{-1}P_n(aq;a,b,c;q).$$

Therefore, the series for $\langle \psi_{aq^2}, \psi_{aq^2}\rangle$ is majorized (up to the finite constant $(1-aq)^{-1}$) by the corresponding series for $\langle \psi_{aq}, \psi_{aq}\rangle$. Thus, $\psi_{aq^2}(x)$ is an eigenfunction of I_2 and the point aq^2 belongs to the spectrum of the operator I_2. Setting $\lambda = aq^2$ in (4.6) and acting similarly, one obtains that $\psi_{aq^3}(x)$ is an eigenfunction of I_2 and the point aq^3 belongs to the spectrum of I_2. Repeating this procedure, one sees that $\psi_{aq^n}(x)$, $n = 1,2,\cdots$, are eigenfunctions of I_2 and the set aq^n, $n = 1,2,\cdots$, belongs to the spectrum of I_2. Likewise, one concludes that $\psi_{cq^n}(x)$, $n = 1,2,\cdots$, are eigenfunctions of I_2 and the set cq^n, $n = 1,2,\cdots$, belongs to the spectrum of I_2. Note that so far we do not know whether the operator I_2 has other spectral points or not. In order to solve this problem we shall proceed as in subsection 3.2.

The functions $\psi_{aq^n}(x)$ and $\psi_{cq^n}(x)$, $n = 1,2,\cdots$, are linearly independent elements of the Hilbert space $\mathcal{H}$. Suppose that aq^n and cq^n, $n = 1,2,\cdots$, constitute the whole spectrum of the operator I_2. Then the set of functions $\psi_{aq^n}(x)$ and $\psi_{cq^n}(x)$, $n = 1,2,\cdots$, is a basis in the space $\mathcal{H}$. Introducing the notations $\Xi_n := \xi_{aq^{n+1}}(x)$ and $\Xi'_n := \xi_{cq^{n+1}}(x)$, $n = 0,1,2,\cdots$, we find from (4.6) that

$$\begin{aligned} J\,\Xi_n &= a^{-1}cq^{-2n-1}(1-aq^{n+1})(1-baq^{n+1}/c)\,\Xi_{n+1} \\ &\quad + d_n\,\Xi_n + a^{-1}cq^{-2n}(1-q^n)(1-aq^n/c)\,\Xi_{n-1}, \end{aligned}$$

$$\begin{aligned} J\,\Xi'_n &= c^{-1}aq^{-2n-1}(1-cq^{n+1})(1-bq^{n+1})\,\Xi_{n+1} \\ &\quad + d'_n\,\Xi_n + c^{-1}aq^{-2n}(1-q^n)(1-cq^n/a)\,\Xi_{n-1}, \end{aligned}$$

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where

$$d_n = \frac{1}{a}[q^{-2n-1}c(1+q) - q^{-n}(ab+ac+a+c)],$$

$$d'_n = \frac{1}{c}[q^{-2n-1}a(1+q) - q^{-n}(ab+ac+a+c)].$$

As we see, the matrix of the operator J in the basis $\Xi_n = \xi_{aq^{n+1}}(x)$, $\Xi'_n = \xi_{cq^{n+1}}(x)$, $n = 0,1,2,\cdots$, is not symmetric, although in the initial basis f_n, $n = 0,1,2,\cdots$, it was symmetric. The reason is that the matrix $M := ((a_{mn})_{m,n=0}^{\infty} \quad (a'_{mn})_{m,n=0}^{\infty})$ with entries

$$a_{mn} := \beta_m(aq^n), \quad a'_{mn} := \beta_m(cq^n), \quad m,n = 0,1,2,\cdots,$$

where $\beta_m(dq^n)$, $d = a, c$, are coefficients (4.3) in the expansion

$$\psi_{dq^n}(x) = \sum_m \beta_m(dq^n)\, f_n(x)$$

(see above), is not unitary. This matrix M is formed by adding the columns of the matrix (a'_{mn}) to the columns of the matrix (a_{mn}) from the right, that is,

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a'_{11} & \cdots & a'_{1l} & \cdots \\ a_{21} & \cdots & a_{2k} & \cdots & a'_{21} & \cdots & a'_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jk} & \cdots & a'_{j1} & \cdots & a'_{jl} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It maps the basis $\{f_n\}$ into the basis $\{\psi_{aq^{n+1}}, \psi_{cq^{n+1}}\}$ in the Hilbert space $\mathcal{H}$. The nonunitarity of the matrix M is equivalent to the statement that the basis $\Xi_n := \xi_{aq^{n+1}}(x)$, $\Xi_n := \xi_{cq^{n+1}}(x)$, $n = 0,1,2,\cdots$, is not normalized. In order to normalize it we have to multiply Ξ_n by appropriate numbers c_n and Ξ'_n by numbers c'_n. Let $\hat{\Xi}_n = c_n \Xi_n$, $\hat{\Xi}'_n = c'_n \Xi_n$, $n = 0,1,2,\cdots$, be a normalized basis. Then the operator J is symmetric in this basis and has the form

$$J\,\hat{\Xi}_n = c_{n+1}^{-1} c_n a^{-1} c q^{-2n-1}(1-aq^{n+1})(1-abq^{n+1}/c)\,\hat{\Xi}_{n+1} + d_n\,\hat{\Xi}_n$$

$$+ c_{n-1}^{-1} c_n a^{-1} c q^{-2n}(1-aq^n/c)(1-q^n)\,\hat{\Xi}_{n-1},$$

$$J\,\hat{\Xi}'_n = {c'}_{n+1}^{-1} {c'}_n c^{-1} a q^{-2n-1}(1-bq^{n+1})(1-cq^{n+1})\,\hat{\Xi}_{n+1} + d'_n\,\hat{\Xi}_n$$

$$+ {c'}_{n-1}^{-1} {c'}_n c^{-1} a q^{-2n}(1-cq^n/a)(1-q^n)\,\hat{\Xi}_{n-1},$$

The symmetricity of the matrix of the operator J in the basis $\{\hat{\Xi}_n, \hat{\Xi}'_n\}$ means that

$$c_{n+1}^{-1} c_n q^{-2n-1}(1-aq^{n+1})(1-abq^{n+1}/c) = c_n^{-1} c_{n+1} q^{-2n-2}(1-aq^{n+1}/c)(1-q^{n+1}),$$

$${c'}_{n+1}^{-1} {c'}_n q^{-2n-1}(1-bq^{n+1})(1-cq^{n+1}) = {c'}_n^{-1} {c'}_{n+1} q^{-2n-2}(1-cq^{n+1}/a)(1-q^{n+1}).$$

that is,

$$\frac{c_n}{c_{n-1}} = \sqrt{q\frac{(1-aq^n)(1-abq^n/c)}{(1-q^n)(1-aq^n/c)}}, \quad \frac{c'_n}{c'_{n-1}} = \sqrt{q\frac{(1-cq^n)(1-bq^n)}{(1-q^n)(1-cq^n/a)}}.$$

Thus,

$$c_n = C\left(q^n\frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n}\right)^{1/2}, \quad c'_n = C'\left(q^n\frac{(bq, cq; q)_n}{(cq/a, q; q)_n}\right)^{1/2},$$

where C and C' are some constants.

Therefore, in the expansions

$$\hat{\psi}_{aq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\,\beta_m(aq^n)\,f^l_m(x) = \sum_m \hat{a}_{mn}\,f^l_m(x), \tag{4.9}$$

$$\hat{\psi}_{cq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c'_n\,\beta_m(cq^n)\,f^l_m(x) = \sum_m \hat{a}'_{mn}\,f^l_m(x), \tag{4.10}$$

the matrix $\hat{M} := ((\hat{a}_{mn})^\infty_{m,n=0}\ \ (\hat{a}'_{mn})^\infty_{m,n=0})$ with entries

$$\hat{a}_{mn} = c_n\,\beta_m(aq^n) = C\left(q^n\frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n}\,\frac{(abq, aq, cq; q)_m\,(1-abq^{2m+1})}{(abq/c, bq, q; q)_m\,(1-abq)(-ac)^m}\right)^{1/2}$$

$$\times\, q^{-m(m+3)/4}\,P_m(aq^{n+1}; a, b, c; q), \tag{4.11}$$

$$\hat{a}'_{mn} = c_n\,\beta_m(cq^n) = C'\left(q^n\frac{(bq, cq; q)_n}{((cq/a, q; q)_n}\,\frac{(abq, aq, cq; q)_m\,(1-abq^{2m+1})}{(abq/c, bq, q; q)_m\,(1-abq)(-ac)^m}\right)^{1/2}$$

$$\times\, q^{-m(m+3)/4}\,P_m(cq^{n+1}; a, b, c; q)\,, \tag{4.12}$$

is unitary, provided that the constants C and C' are appropriately chosen. In order to calculate these constants, one can use the relations

$$\sum_{m=0}^\infty |\hat{a}_{mn}|^2 = 1\,, \quad \sum_{m=0}^\infty |\hat{a}'_{mn}|^2 = 1\,,$$

for $n = 0$. Then these sums are multiples of the sums in (4.7) and (4.8), so we find that

$$C = \frac{(bq, cq; q)^{1/2}_\infty}{(abq^2, c/a; q)^{1/2}_\infty}, \quad C' = \frac{(aq, abq/c; q)^{1/2}_\infty}{(abq^2, a/c; q)^{1/2}_\infty}. \tag{4.13}$$

The coefficients c_n and c'_n in (4.9)–(4.12) are thus real and equal to

$$c_n = \left(\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty}\,\frac{(abq/c, aq; q)_n\,q^n}{(aq/c, q; q)_n}\right)^{1/2},$$
$$c'_n = \left(\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty}\,\frac{(bq, cq; q)_n\,q^n}{(cq/a, q; q)_n}\right)^{1/2}.$$

The orthogonality of the matrix $\hat{M} \equiv ((\hat{a}_{mn})_{m,n=0}^{\infty}\ \ (\hat{a}'_{mn})_{m,n=0}^{\infty})$ means that

$$\sum_m \hat{a}_{mn}\hat{a}_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}'_{mn}\hat{a}'_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}_{mn}\hat{a}'_{mn'} = 0, \tag{4.14}$$

$$\sum_n (\hat{a}_{mn}\hat{a}_{m'n} + \hat{a}'_{mn}\hat{a}'_{m'n}) = \delta_{mm'}. \tag{4.15}$$

Substituting the expressions for $\hat{a}_{mn}$ and $\hat{a}'_{mn}$ into (4.15), one obtains the relation

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n\, q^n}{(aq/c, q; q)_n} P_m(aq^{n+1})\, P_{m'}(aq^{n+1})$$

$$+\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n\, q^n}{(cq/a, q; q)_n} P_m(cq^{n+1})\, P_{m'}(cq^{n+1})$$

$$= \frac{(1-abq)(bq, abq/c, q; q)_m}{(1-abq^{2m+1})(aq, abq, cq; q)_m} (-ac)^m\, q^{m(m+3)/2}\, \delta_{mm'}\,. \tag{4.16}$$

This identity must give an orthogonality relation for the big q-Jacobi polynomials $P_m(y) \equiv P_m(y; a, b, c; q)$. An only gap, which appears here, is the following. We have assumed that the points aq^n and cq^n, $n = 0, 1, 2, \cdots$, exhaust the whole spectrum of the operator I_2. As in the case of the operator I_1 in subsection 3.2, if the operator I_2 had other spectral points x_k, then on the left-hand side of (4.16) would appear other summands $\mu_{x_k} P_m(x_k; a, b, c; q) P_{m'}(x_k; a, b, c; q)$, which correspond to these additional points. Let us show that these additional summands do not appear. We set $m = m' = 0$ in the relation (4.16) with the additional summands. This results in the equality

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n q^n}{(aq/c, q; q)_n}$$

$$+\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} + \sum_k \mu_{x_k} = 1. \tag{4.17}$$

In order to show that $\sum_k \mu_{x_k} = 0$, take into account the relation

$$\frac{(Aq/C, Bq/C; q)_\infty}{(q/C, ABq/C; q)_\infty}\, {}_2\phi_1(A, B; C;\ q, q)$$

$$+ \frac{(A, B; q)_\infty}{(C/q, ABq/C; q)_\infty}\, {}_2\phi_1(Aq/C, Bq/C; q^2/C;\ q, q) = 1$$

(see formula (2.10.13) in [21]). Putting here $A = aq$, $B = abq/c$ and $C = aq/c$, we obtain relation (4.17) without the summand $\sum_k \mu_{x_k}$. Therefore, in (4.17) the sum $\sum_k \mu_{x_k}$ does really vanish and formula (4.16) gives an orthogonality relation for big q-Jacobi polynomials.

By using the operators I_2 and J, we thus derived the orthogonality relation for big q-Jacobi polynomials.

The orthogonality relation (4.16) enables one to formulate the following statement: *The spectrum of the operator I_2 coincides with the set of points aq^{n+1} and cq^{n+1}, $n = 0, 1, 2, \cdots$. The spectrum is simple and has one accumulation point at 0.*

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4.3. Dual big q-Jacobi polynomials. Now we consider the relations (4.14). They give the orthogonality relation for the set of matrix elements $\hat{a}_{mn}$ and $\hat{a}'_{mn}$, viewed as functions of m. Up to multiplicative factors, they coincide with the functions

$$F_n(x;a,b,c;q) := {}_3\phi_2(x, abq/x, aq^{n+1};\; aq, cq;\; q, q), \;\; n = 0, 1, 2, \cdots, \tag{4.18}$$

$$F'_n(x;a,b,c;q) := {}_3\phi_2(x, abq/x, cq^{n+1};\; aq, cq;\; q, q) \tag{4.19}$$

$$\equiv F_n(x;c,ab/c,a), \;\; n = 0, 1, 2, \cdots,$$

considered on the corresponding sets of points. Namely, we have

$$\hat{a}_{mn} \equiv \hat{a}_{mn}(a,b,c) = C \left(q^n \frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n} \frac{(abq, aq, cq; q)_m \, (1 - abq^{2m+1})}{(abq/c, bq, q; q)_m \, (1 - abq)(-ac)^m} \right)^{1/2}$$

$$\times \, q^{-m(m+3)/4} \, F_n(q^{-m}; a, b, c; q),$$

$$\hat{a}'_{mn} \equiv \hat{a}'_{mn}(a,b,c) = C' \left(q^n \frac{(bq, cq; q)_n}{(cq/a, q; q)_n} \frac{(abq, aq, cq; q)_m \, (1 - abq^{2m+1})}{(abq/c, bq, q; q)_m \, (1 - abq)(-ac)^m} \right)^{1/2}$$

$$\times \, q^{-m(m+3)/4} \, F'_n(q^{-m}; a, b, c; q) \equiv \hat{a}_{mn}(c, ab/c, a),$$

where C and C' are given by formulas (4.13). The relations (4.14) lead to the following orthogonality relations for the functions (4.18) and (4.19):

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{m=0}^{\infty} \rho(m) F_n(q^{-m}; a, b, c; q) \, F_{n'}(q^{-m}; a, b, c; q)$$

$$= \frac{(aq/c, q; q)_n}{(aq, abq/c; q)_n q^n} \delta_{nn'}, \tag{4.20}$$

$$\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{m=0}^{\infty} \rho(m) \, F'_n(q^{-m}; a, b, c; q) \, F'_{n'}(q^{-m}; a, b, c; q)$$

$$= \frac{(cq/a, q; q)_n}{(bq, cq; q)_n q^n} \, \delta_{nn'}, \tag{4.21}$$

$$\sum_{m=0}^{\infty} \rho(m) \, F_n(q^{-m}; a, b, c; q) \, F'_{n'}(q^{-m}; a, b, c; q) = 0, \tag{4.22}$$

where

$$\rho(m) := \frac{(1 - abq^{2m+1})(aq, abq, cq; q)_m}{(1 - abq)(bq, abq/c, q; q)_m \, (-ac)^m} \, q^{-m(m+3)/2}.$$

There is another form for the functions $F_n(q^{-m};a,b,c;q)$ and $F'_n(q^{-m};a,b,c;q)$. Indeed, one can use the relation (III.12) of Appendix III in [21] to obtain that

$$F_n(q^{-m};a,b,c;q) = \frac{(cq^{-m}/ab;q)_m}{(cq;q)_m}(abq^{m+1})^m \, {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| \, q, aq^{n+1}/c \right)$$

$$= \frac{(abq/c;q)_m}{(cq;q)_m}(-c)^m q^{m(m+1)/2} \, {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| \, q, aq^{n+1}/c \right)$$

and

$$F'_n(q^{-m};a,b,c;q) = \frac{(bq;q)_m}{(aq;q)_m}(-a)^m q^{m(m+1)/2} \, {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ bq, cq \end{matrix} \middle| \, q, cq^{n+1}/a \right).$$

The basic hypergeometric functions ${}_3\phi_2$ in these formulas are polynomials in $\mu(m) := q^{-m} + ab\,q^{m+1}$. So if we introduce the notation

$$D_n(\mu(m);a,b,c|q) := {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| \, q, aq^{n+1}/c \right), \tag{4.23}$$

then

$$F_n(q^{-m};a,b,c;q) = \frac{(abq/c;q)_m}{(cq;q)_m}(-c)^m q^{m(m+1)/2} D_n(\mu(m);a,b,c|q),$$

$$F'_n(q^{-m};a,b,c;q) = \frac{(bq;q)_m}{(aq;q)_m}(-a)^m q^{m(m+1)/2} D_n(\mu(m);b,a,ab/c|q).$$

Formula (4.20) directly leads to the orthogonality relation for the polynomials $D_n(\mu(m)) \equiv D_n(\mu(m);a,b,c|q)$:

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(aq,abq,abq/c;q)_m}{(1-abq)(bq,cq,q;q)_m} \, (-c/a)^m \, q^{m(m-1)/2} \, D_n(\mu(m)) \, D_{n'}(\mu(m))$$

$$= \frac{(abq^2,c/a;q)_\infty}{(bq,cq;q)_\infty} \frac{(aq/c,q;q)_n}{(aq,abq/c;q)_n q^n} \, \delta_{nn'}. \tag{4.24}$$

From (4.21) one obtains the orthogonality relation for the polynomials $D_n(\mu(m);b,a,ab/c|q)$ (which follows also from the relation (4.24) by interchanging a and b and replacing c by ab/c).

We call the polynomials $D_n(\mu(m);a,b,c|q)$ *dual big q-Jacobi polynomials*. It is natural to ask whether they can be identified with some known and thoroughly studied set of polynomials. The answer is: they can be obtained from the q-Racah polynomials $R_n(\mu(x);a,b,c,d|q)$ of Askey and Wilson [14] by setting $a = q^{-N-1}$ and sending $N \to \infty$, that is,

$$D_n(\mu(x);a,b,c|q) = \lim_{N\to\infty} R_n(\mu(x);q^{-N-1},a/c,a,b|q).$$

Observe that the orthogonality relation (4.24) can be also derived from formula (4.12) in [30]. But the derivation of this formula (4.12) is rather complicated.

The dual polynomials (4.23) and the big q-Jacobi polynomials (4.2) are interrelated in the following way:

$$D_n(\mu(m); a, b, c|q) = \frac{(cq;q)_m}{(abq/c;q)_m}(-c)^{-m}q^{-m(m+1)/2}P_m(aq^{n+1}; a, b, c|q).$$

It is worth noting here that in the limit as $c \to 0$ the dual big q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ coincide with the dual little q-Jacobi polynomials $d_n(\mu(x); b, a|q)$, defined in section 3. The dual little q-Jacobi polynomials $d_n(\mu(x); a, b|q)$ reduce, in turn, to the Al-Salam–Carlitz II polynomials $V_n^{(a)}(s;q)$ on the q-linear lattice $s = q^{-x}$ (see [27], p. 114) in the case when the parameter b vanishes, that is,

$$d_n(\mu(x); a, 0|q) = {}_2\phi_0(q^{-n},\, q^{-x}; -;\; q, q^n/a) = (-a)^{-n}q^{n(n-1)/2}\, V_n^{(a)}(q^{-x};q).$$

This means that we have now a complete chain of reductions

$$R_n(\mu(x); a, b, c, d|q) \underset{a\to\infty}{\longrightarrow} D_n(\mu(x); c, d, c/b|q) \underset{b\to\infty}{\longrightarrow} d_n(\mu(x); d, c|q) \underset{c=0}{\longrightarrow} V_n^{(d)}(q^{-x};q)$$

from the four-parameter family of q-Racah polynomials, which occupy the upper level in the Askey-scheme of basic hypergeometric polynomials (see [27], p. 62), down to the one-parameter set of Al-Salam–Carlitz II polynomials from the second level in the same scheme. So, the dual big and dual little q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ and $d_n(\mu(x); a, b|q)$ should occupy the fourth and third level in the Askey-scheme, respectively.

The recurrence relations for the polynomials $D_n(\mu(m) \equiv D_n(\mu(m); a, b, c|q)$ are obtained from the q-difference equation (4.5). It has the form

$$\begin{aligned}(q^{-m} - 1)&(1 - abq^{m+1})D_n(\mu(m)) = \\ &A_nD_{n+1}(\mu(m)) - (A_n + C_n)D_n(\mu(m)) + C_nD_{n-1}(\mu(m)),\end{aligned}$$

where

$$A_n = q^{-2n-1}(1 - aq^{n+1})\left[(c/a) - bq^{n+1}\right], \quad C_n = q^{-2n}(1 - q^n)\left[(c/a) - q^n\right].$$

The relation (4.22) leads to the equality (another proof of this relation is given in Appendix)

$$\begin{aligned}&\sum_{m=0}^{\infty}(-1)^m\frac{(1 - abq^{2m+1})(abq;q)_m}{(1 - abq)(q;q)_m}\, q^{m(m-1)/2} \\ &\times D_n(\mu(m); a, b, c|q)\, D_{n'}(\mu(m); b, a, ab/c|q) = 0. \qquad (4.25)\end{aligned}$$

Note that from the expression (4.23) for the dual big q-Jacobi polynomials it follows that they possess the symmetry property

$$D_n(\mu(m); a, b, c|q) = D_n(\mu(m); ab/c, c, b|q). \qquad (4.26)$$

The set of functions (4.18) and (4.19) form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$ of functions, defined on the set of points $m = 0, 1, 2, \cdots$, with the scalar product

$$\langle f_1, f_2\rangle = \sum_{m=0}^{\infty}\rho(m)\, f_1(m)\, \overline{f_2(m)},$$

where $\rho(m)$ is the same as in formulas (4.20)–(4.22). Consequent from this fact, one can deduce (in the same way as in the case of dual little q-Jacobi polynomials) that *the dual big q-Jacobi polynomials $D_n(\mu(m); a, b, c|q)$ correspond to indeterminate moment problem and the orthogonality measure for them, given by formula* (4.24)*, is not extremal.*

It is difficult to find extremal measures for these polynomials.

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4.4. Generating functions for dual big q-Jacobi polynomials. Generating functions are known to be of great importance in the theory of orthogonal polynomials (see, for example, [3] and [33]). For the sake of completeness, we briefly discuss in this section some instances of linear generating functions for the dual q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ and $d_n(\mu(x); a, b|q)$. To start with, let us consider a generating-function formula

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} t^n D_n(\mu(x); a, b, c|q) = \frac{(aqt;q)_\infty}{(t;q)_\infty} {}_2\phi_2\left(\begin{matrix} q^{-x},\ abq^{x+1} \\ abq/c,\ aqt \end{matrix} \middle| q,\ aqt/c \right), \tag{4.27}$$

where $|t| < 1$ and, as before, $\mu(x) = q^{-x} + abq^{x+1}$. To verify (4.27), insert the explicit form (4.23) of the dual big q-Jacobi polynomials

$$D_n(\mu(x); a, b, c|q) = \sum_{k=0}^{n} \frac{(q^{-x}, abq^{x+1}, q^{-n}; q)_k}{(aq, abq/c, q; q)_k} \left(\frac{aq^{n+1}}{c} \right)^k$$

into the left side of (4.27) and interchange the order of summation. The subsequent use of the relations

$$(a;q)_{m+k} = (a;q)_m (aq^m;q)_k = (a;q)_k (aq^k;q)_m,$$

$$(q^{-m-k};q)_k = (-1)^k q^{-mk-k(k+1)/2} (q^{m+1};q)_k$$

(see [21], Appendix I) simplifies the inner sum and enables one to evaluate it by the q-binomial formula (3.13). This gives the quotient of two infinite products in front of ${}_2\phi_2$ on the right side of (4.27), times $(aqt;q)_k^{-1}$. The remaining sum over k yields ${}_2\phi_2$ series itself.

As a consistency check, one may also obtain (4.27) directly from the generating function for the q-Racah polynomials $R_n(\mu(x); \alpha, \beta, \gamma, \delta|q)$ (see formula (3.2.13) in [27]) by setting $\alpha = q^{-N-1}$ and sending $N \to \infty$. This results in the relation

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n}\ t^n\ D_n(\mu(x); a, b, c|q)$$

$$= \frac{(aq^{x+1}t;q)_\infty}{(t;q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-x},\ c^{-1}q^{-x} \\ abq/c \end{matrix} \middle| q,\ atq^{x+1} \right). \tag{4.28}$$

The left side of (4.28) depends on the variable x by dint of the combination $\mu(x) = q^{-x} + ab\,q^{x+1}$. Off hand, it is not evident that the right side of (4.28) is also a function of the lattice $\mu(x)$. Nevertheless, this is the case. Moreover, the right sides of (4.27) and (4.28) are equivalent: this fact is known in the theory of special functions as Jackson's transformation (see, for example, [21])

$${}_2\phi_1(a, b;\ c;\ q, z) = \frac{(az;q)_\infty}{(z;q)_\infty}\ {}_2\phi_2(a, c/b;\ c, az;\ q, bz)\,.$$

The symmetry property (4.26) of the dual big q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$, combined with (4.27), generates another relation

$$\sum_{n=0}^{\infty} \frac{(abq/c;q)_n}{(q;q)_n} t^n\ D_n(\mu(x); a, b, c|q) = \frac{(abqt/c;q)_\infty}{(t;q)_\infty} {}_2\phi_2\left(\begin{matrix} q^{-x},\ abq^{x+1} \\ aq,\ abqt/c \end{matrix} \middle| q,\ aqt/c \right)$$

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$$= \frac{(abtq^{x+1}/c;q)_\infty}{(t;q)_\infty}\, {}_2\phi_1 \left(\begin{matrix} q^{-x},\ b^{-1}q^{-x} \\ aq \end{matrix} \middle|\, q,\ abtq^{x+1}/c \right).$$

Similarly, a generating function for the dual little q-Jacobi polynomials has the form

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n}{(q;q)_n}\,(at)^n\, d_n(\mu(x);a,b|q) = \frac{(tq^{-x}, abtq^{x+1};q)_\infty}{(at,t;q)_\infty}. \tag{4.29}$$

One can verify (4.29) directly by inserting the explicit form (3.17) of $d_n(\mu(x);a,b|q)$ into the left side of (4.29) and repeating the same steps as in the case of deriving (4.27). This will lead to the expression

$$\frac{(abqt;q)_\infty}{(at;q)_\infty}\, {}_2\phi_1(q^{-x},\ abq^{x+1};\ abqt;\ q,t)$$

and it remains only to employ Heine's summation formula (1.5.1) from [21]. After a simple re-scaling of the parameters the generating function (4.29) coincides with that, obtained earlier in [17].

The simplest way of obtaining (4.29) is to send $c \to 0$ in both sides of (4.28): the ${}_2\phi_1$ series on the right side of (4.28) reduces to ${}_1\phi_0(q^{-x};-;\ q,t/a)$, which is evaluated by the q-binomial formula (3.13).

Finally, when the parameter b vanishes, (4.29) reduces to the known generating function

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, q^{n(n-1)/2}}{(q;q)_n}\, t^n\, V_n^{(a)}(q^{-x};q) = \frac{(tq^{-x};q)_\infty}{(at,t;q)_\infty}$$

for the Al-Salam–Carlitz II polynomials (see (3.25.11) in [27]).

5. Discrete q-ultraspherical polynomials and their duals.

5.1. Discrete q-ultraspherical polynomials.

For the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ the following limit relation holds:

$$\lim_{q\uparrow 1} P_n(x;q^\alpha,q^\beta,-q^\gamma;q) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)},$$

where γ is real. Therefore, $\lim_{q\uparrow 1} P_n(x;q^\alpha,q^\alpha,-q^\gamma;q)$ is a multiple of the Gegenbauer (ultraspherical) polynomial $C_n^{(\alpha-1/2)}(x)$. For this reason, we introduce the notation

$$C_n^{(a^2)}(x;q) := P_n(x;a,a,-a;q) = {}_3\phi_2(q^{-n},a^2q^{n+1},x;\ aq,-aq;\ q,q). \tag{5.1}$$

It is obvious from (5.1) that $C_n^{(a)}(x;q)$ is a rational function in the parameter a.

From the recurrence relation for the big q-Jacobi polynomials (see subsection 4.1) one readily verifies that the polynomials (5.1) satisfy the following three-term recurrence relation:

$$x\, C_n^{(a)}(x;q) = A_n(a)\, C_{n+1}^{(a)}(x;q) + C_n(a)\, C_{n-1}^{(a)}(x;q), \tag{5.2}$$

where $A_n(a) = (1-aq^{n+1})/(1-aq^{2n+1})$, $C_n(a) = 1 - A_n(a)$, and $C_0^{(a)}(x;q) \equiv 1$.

An orthogonality relation for $C_n^{(a)}(x;q)$, which follows from that for the big q-Jacobi polynomials and is considered in the next section, holds for positive values of a. We shall see

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that the polynomials $C_n^{(a)}(x;q)$ are orthogonal also for imaginary values of a and x. In order to dispense with imaginary numbers in this case, we introduce the following notation:

$$\tilde{C}_n^{(a^2)}(x;q) := (-\mathrm{i})^n\, C_n^{(-a^2)}(\,\mathrm{i}x;q) = (-\mathrm{i})^n\, P_n(\mathrm{i}x;\mathrm{i}a,\mathrm{i}a,-\mathrm{i}a;q),$$

where x is real and $0<a<\infty$. The polynomials $\tilde{C}_n^{(a)}(x;q)$ satisfy the recurrence relation

$$x\,\tilde{C}_n^{(a)}(x;q) = \tilde{A}_n(a)\,\tilde{C}_{n+1}^{(a)}(x;q) + \tilde{C}_n(a)\,\tilde{C}_{n-1}^{(a)}(x;q), \tag{5.3}$$

where $\tilde{A}_n(a) = A_n(-a) = (1+aq^{n+1})/(1+aq^{2n+1})$, $\tilde{C}_n(a) = \tilde{A}_n(a)-1$, and $\tilde{C}_0^{(a)}(x;q)\equiv 1$. Note that $\tilde{A}_n(a)\ge 1$ and, hence, coefficients in the recurrence relation (5.3) for $\tilde{C}_n^{(a)}(x;q)$ satisfy the conditions $\tilde{A}_n(a)\tilde{C}_{n+1}(a) > 0$ of Favard's characterization theorem for $n = 0,1,2,\cdots$ (see, for example, [21]). This means that these polynomials are orthogonal with respect to a positive measure with infinitely many points of support. An explicit form of this measure is derived in the next section.

So, we have

$$\tilde{C}_n^{(a)}(x;q) = (-\mathrm{i})^n\, C_n^{(-a)}(\mathrm{i}x;q) = (-\mathrm{i})^n\, {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, -aq^{n+1}, \mathrm{i}x \\ \mathrm{i}\sqrt{a}q, -\mathrm{i}\sqrt{a}q \end{matrix}\right| q,q\right). \tag{5.4}$$

(Here and everywhere below under $\sqrt{a}$, $a>0$, we understand a positive value of the root.) From the recurrence relation (5.3) it follows that the polynomials (5.4) are real for $x\in\mathbb{R}$ and $0<a<\infty$. From (5.4) it is also obvious that they are rational functions in the parameter a. Observe that the situation when along with orthogonal polynomials $p_n(x)$, depending on some parameters, the set of polynomials $(-\mathrm{i})^n\, p_n(\mathrm{i}x)$ is also orthogonal (but for other values of parameters) is known; see, for example, [29], [13], and [18].

We show below that the polynomials $C_n^{(a)}(x;q)$ and $\tilde{C}_n^{(a)}(x;q)$, interrelated by (5.4), are orthogonal with respect to discrete measures. For this reason, they may be regarded [9] as a discrete version of q-ultraspherical polynomials of Rogers (see, for example, [4]).

PROPOSITION 5.1. *The following expressions for the discrete q-ultraspherical polynomials* (5.1) *hold:*

$$C_{2k}^{(a)}(x;q) = \frac{(q;q^2)_k\, a^k}{(aq^2;q^2)_k}(-1)^k q^{k(k+1)} p_k(x^2/aq^2;q^{-1},a|q^2), \tag{5.5}$$

$$C_{2k+1}^{(a)}(x;q) = \frac{(q^3;q^2)_k\, a^k}{(aq^2;q^2)_k}(-1)^k q^{k(k+1)}\, x\, p_k(x^2/aq^2;q,a|q^2), \tag{5.6}$$

where $p_k(y;a,b|q)$ are the little q-Jacobi polynomials (3.3).

Proof. To start with (5.5), apply Singh's quadratic transformation for a terminating ${}_3\phi_2$ series

$${}_3\phi_2\left(\left.\begin{matrix} a^2,\, b^2,\, c \\ abq^{1/2}, -abq^{1/2} \end{matrix}\right| q,q\right) = {}_3\phi_2\left(\left.\begin{matrix} a^2,\, b^2,\, c^2 \\ a^2b^2q,\ 0 \end{matrix}\right| q^2,q^2\right), \tag{5.7}$$

which is valid when both sides in (5.7) terminate (see [21], formula (3.10.13)), to the expression in (5.1) for q-ultraspherical polynomials $C_{2k}^{(a)}(x;q)$. This results in the following:

$$C_{2k}^{(a)}(x;q) = {}_3\phi_2\left(q^{-2k},\ aq^{2k+1},\ x^2;\ aq^2,\ 0;\ q^2,q^2\right).$$

Now apply to this basic hypergeometric series ${}_3\phi_2$ the transformation formula

$$(5.8)\qquad {}_2\phi_1\left(\begin{matrix} q^{-n},\ b \\ c \end{matrix}\,\middle|\, q, z\right) = \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},\ b,\ bzq^{-n}/c \\ bq^{1-n}/c,\ 0 \end{matrix}\,\middle|\, q, q\right)$$

(see formula (III.7) from Appendix III in [21]) in order to get

$$C_{2k}^{(a)}(x;q) = \frac{(q;q^2)_k\, a^k}{(aq^2;q^2)_k}(-1)^k q^{k(k+1)}\, {}_2\phi_1\left(q^{-2k},\ aq^{2k+1};\ q;\ q^2, x^2/a\right).$$

Comparing this formula with the expression for the little q-Jacobi polynomials (3.3) one arrives at (5.5).

One can now prove (5.7) by induction with the aid of (5.8) and the recurrence relation (5.2). Indeed, since $C_0^{(a)}(x;q) \equiv 1$ and $A_0(a) = 1$, one obtains from (5.2) that $C_1^{(a)}(x;q) = x$. As the next step use the fact that $C_2^{(a)}(x;q) = {}_3\phi_2(q^{-2}, aq^3, x^2;\ aq^2, 0;\ q^2, q^2)$ to evaluate from (5.2) explicitly that

$$C_3^{(a)}(x;q) = x\, {}_3\phi_2\left(q^{-2},\ aq^5,\ x^2;\ aq^2,\ 0;\ q^2,\ q^2\right).$$

So, let us suppose that

$$(5.9)\qquad C_{2k-1}^{(a)}(x;q) = x\, {}_3\phi_2\left(q^{-2(k-1)},\ aq^{2k+1},\ x^2;\ aq^2,\ 0;\ q^2,\ q^2\right)$$

for $k = 1, 2, 3, \cdots$, and evaluate a sum $A_{2k}^{-1}(a)x\, C_{2k}^{(a)}(x;q) + (1 - A_{2k}^{-1}(a))C_{2k-1}^{(a)}(x;q)$. As follows from the recurrence relation (5.2), this sum should be equal to $C_{2k+1}^{(a)}(x;q)$. This is the case because it is equal to

$$x\left\{A_{2k}^{-1}\, {}_3\phi_2\left(\begin{matrix} q^{-2k}, aq^{2k+1}, x^2 \\ aq^2, 0 \end{matrix}\,\middle|\, q^2, q^2\right) + (1 - A_{2k}^{-1})\, {}_3\phi_2\left(\begin{matrix} q^{-2(k-1)}, aq^{2k+1}, x^2 \\ aq^2, 0 \end{matrix}\,\middle|\, q^2, q^2\right)\right\}$$

$$(5.10)\qquad = x\, {}_3\phi_2\left(\begin{matrix} q^{-2k},\ aq^{2k+3},\ x^2 \\ aq^2,\ 0 \end{matrix}\,\middle|\, q^2, q^2\right),$$

if one takes into account readily verified identities

$$A_{2k}^{-1}(q^{-2k};q^2)_m + (1 - A_{2k}^{-1})\,(q^{-2(k-1)};q^2)_m = \frac{1 - aq^{2(k+m)+1}}{1 - aq^{2k+1}}\,(q^{-2k};q^2)_m,$$

$$\frac{1 - aq^{2(k+m)+1}}{1 - aq^{2k+1}}\,(aq^{2k+1};q^2)_m = (aq^{2k+3};q^2)_m,$$

for $m = 0, 1, 2, \cdots, k$. The right side of (5.10) does coincide with $C_{2k+1}^{(a)}(x;q)$, defined by the same expression (5.9) with $k \to k+1$. Thus, it remains only to apply the same transformation formula (5.8) in order to arrive at (5.6). Proposition is proved. $\square$

Remark. Observe that in the process of proving formula (5.6), we established a quadratic transformation

$$(5.11)\qquad {}_3\phi_2\left(\begin{matrix} q^{-2k-1}, aq^{2k+2}, x \\ \sqrt{a}q, -\sqrt{a}q \end{matrix}\,\middle|\, q, q\right) = x\, {}_3\phi_2\left(\begin{matrix} q^{-2k}, aq^{2k+3}, x^2 \\ aq^2,\ 0 \end{matrix}\,\middle|\, q^2, q^2\right)$$

for the terminating basic hypergeometric polynomials ${}_3\phi_2$ with $k = 0, 1, 2, \cdots$. The left side in (5.11) defines the polynomials $C_{2k+1}^{(a)}(x; q)$ by (5.1), whereas the right side follows from the expression (5.10) for the same polynomials. The formula (5.11) represents an extension of Singh's quadratic transformation (5.7) to the case when $a^2 = q^{-2k-1}$ and, therefore, the left side in (5.7) terminates, but the right side does not.

It follows from (5.4)–(5.6) that

$$\tilde{C}_{2k}^{(a)}(x;q) = \frac{(q;q^2)_k\, a^k}{(-aq^2;q^2)_k}(-1)^k\, q^{k(k+1)}\, p_k(x^2/aq^2; q^{-1}, -a|q^2), \tag{5.12}$$

$$\tilde{C}_{2k+1}^{(a)}(x;q) = \frac{(q^3;q^2)_k\, a^k}{(-aq^2;q^2)_k}\,(-1)^k\, q^{k(k+1)} x\, p_k(x^2/aq^2; q, -a|q^2). \tag{5.13}$$

In particular, it is clear from these formulas that the polynomials $\tilde{C}_n^{(a)}(x;q)$ are real-valued for $x \in \mathbb{R}$ and $a > 0$.

5.2. Orthogonality relations for discrete q-ultraspherical polynomials. Since the polynomials $C_n^{(a)}(x;q)$ are a particular case of the big q-Jacobi polynomials, an orthogonality relation for them follows from (4.16). Setting $a = b = -c$, $a > 0$, into (4.16) and considering the case when $m = 2k$ and $m' = 2k'$, one verifies that two sums on the left of (4.16) coincide (since $ab/c = -a = c$) and we obtain the following orthogonality relation for $C_{2k}^{(a)}(x;q)$:

$$\sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\; C_{2k}^{(a)}(\sqrt{a}q^{s+1};q)\, C_{2k'}^{(a)}(\sqrt{a}q^{s+1};q)$$

$$= \frac{(aq^3;q^2)_\infty}{(q;q^2)_\infty}\,\frac{(1-aq)a^{2k}}{1-aq^{4k+1}}\,\frac{(q;q)_{2k}\, q^{k(2k+3)}}{(aq;q)_{2k}}\,\delta_{kk'},$$

where $\sqrt{a}$, $a > 0$, denotes a positive value of the root. Thus, *the family of polynomials* $C_{2k}^{(a)}(x;q)$, $k = 0, 1, 2, \cdots$, *with* $0 < a < q^{-2}$, *is orthogonal on the set of points* $\sqrt{a}q^{s+1}$, $s = 0, 1, 2, \cdots$.

As we know, the polynomials $C_{2k}^{(a)}(x;q)$ are functions in x^2, that is, $C_{2k}^{(a)}(\sqrt{a}q^{s+1};q)$ is in fact a function in aq^{2s+2}. The set of functions $C_{2k}^{(a)}(x;q)$, $k = 0, 1, 2, \cdots$, constitutes a complete basis in the Hilbert space $\mathfrak{l}^2$ of functions $f(x^2)$ with the scalar product

$$(f_1, f_2) = \sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s} f_1(aq^{2s+2})\overline{f_2(aq^{2s+2})}.$$

This result can be obtained from the orthogonality relation for the little q-Jacobi polynomials, if one takes into account formula (5.5).

Putting $a = b = -c$, $a > 0$, into (4.16) and considering the case when $m = 2k+1$ and $m' = 2k'+1$, one verifies that two sums on the left of (4.16) again coincide and we obtain the following orthogonality relation for $C_{2k+1}^{(a)}(x;q)$:

$$\sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, C_{2k+1}^{(a)}(\sqrt{a}\, q^{s+1};q)\, C_{2k'+1}^{(a)}(\sqrt{a}\, q^{s+1};q)$$

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$$= \frac{(aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1-aq)\,a^{2k+1}}{(1-aq^{4k+3})} \frac{(q;q)_{2k+1}}{(aq;q)_{2k+1}}\, q^{(k+2)(2k+1)}\, \delta_{kk'}.$$

The polynomials $C^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, *with* $0<a<q^{-2}$, *are thus orthogonal on the set of points* $\sqrt{a}\,q^{s+1}$, $s=0,1,2,\cdots$.

The polynomials $x^{-1}C^{(a)}_{2k+1}(x;q)$ are functions in x^2, that is, $x^{-1}C^{(a)}_{2k+1}(\sqrt{a}\,q^{s+1};q)$ are in fact functions in aq^{2s+2}. The collection of functions $C^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, constitute a complete basis in the Hilbert space $\mathfrak{l}^2$ of functions of the form $F(x)=xf(x^2)$ with the scalar product

$$(F_1,F_2)=\sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, F_1(\sqrt{a}\,q^{s+1})\,\overline{F_2(\sqrt{a}\,q^{s+1})}.$$

Again, this result can be obtained from the orthogonality relation for the little q-Jacobi polynomials if one takes into account formula (5.6).

We have shown that the polynomials $C^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, as well as the polynomials $C^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, are orthogonal on the set of points $\sqrt{a}\,q^{s+1}$, $s=0,1,2,\cdots$. However, the polynomials $C^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, are not orthogonal to the polynomials $C^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, on this set of points. In order to obtain an orthogonality for the whole collection of the polynomials $C^{(a)}_{n}(x;q)$, $n=0,1,2,\cdots$, one has to consider them on the set of points $\pm\sqrt{a}\,q^{s+1}$, $s=0,1,2,\cdots$. Since the polynomials from the first set are even and the polynomials from the second set are odd, for each $k,k'\in\{0,1,2,\cdots\}$ the infinite sum

$$I_1 \equiv \sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, C^{(a)}_{2k}(\sqrt{a}\,q^{s+1};q)\, C^{(a)}_{2k'+1}(\sqrt{a}\,q^{s+1};q)$$

coincides with the following one

$$I_2 \equiv -\sum_{s=0}^{\infty} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, C^{(a)}_{2k}(-\sqrt{a}\,q^{s+1};q)\, C^{(a)}_{2k'+1}(-\sqrt{a}\,q^{s+1};q).$$

Therefore, $I_1-I_2=0$. This gives the orthogonality of polynomials from the set $C^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, with respect to the polynomials from the set $C^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$. The orthogonality relation for the whole set of polynomials $C^{(a)}_{n}(x;q)$, $n=0,1,2,\cdots$, can be written in the form

$$\sum_{s=0}^{\infty}\sum_{\varepsilon=\pm1} \frac{(aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, C^{(a)}_{n}(\varepsilon\sqrt{a}\,q^{s+1};q)\, C^{(a)}_{n'}(\varepsilon\sqrt{a}\,q^{s+1};q)$$

$$= \frac{(aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1-aq)\,a^{n}}{(1-aq^{2n+1})} \frac{(q;q)_{n}\, q^{n(n+3)/2}}{(aq;q)_{n}}\, \delta_{nn'}.$$

We thus see that *the polynomials* $C^{(a)}_{n}(x;q)$, $n=0,1,2,\cdots$, *with* $0<a<q^{-2}$ *are orthogonal on the set of points* $\pm\sqrt{a}\,q^{s+1}$, $s=0,1,2,\cdots$.

An orthogonality relation for the polynomials $\tilde{C}^{(a)}_{n}(x;q)$, $n=0,1,2,\cdots$, is derived by using the relations (5.12), (5.13), and the orthogonality relation for the little q-Jacobi polynomials. Writing down the orthogonality relation (3.12) for the polynomials $p_k(x^2/aq^2;q^{-1},$

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$-a|q^2)$ and using the relation (5.12), one finds an orthogonality relation for the set of polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, with $a>0$. It has the form

$$\sum_{s=0}^{\infty} \frac{(-aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, \tilde{C}^{(a)}_{2k}(\sqrt{a}\, q^{s+1};q)\, \tilde{C}^{(a)}_{2k'}(\sqrt{a}\, q^{s+1};q)$$

$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1+aq)\, a^{2k}}{(1+aq^{4k+1})} \frac{(q;q)_{2k}}{(-aq;q)_{2k}}\, q^{k(2k+3)}\, \delta_{kk'}.$$

Consequently, *the family of polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, is orthogonal on the set of points $\sqrt{a}\, q^{s+1}$, $s=0,1,2,\cdots$.*

As in the case of polynomials $C^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, the set $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, is complete in the Hilbert space of functions $f(x^2)$ with the corresponding scalar product.

Similarly, using formula (5.13) and the orthogonality relation for the little q-Jacobi polynomials $p_k(x^2/aq^2;q,-a|q^2)$, we find the orthogonality relation

$$\sum_{s=0}^{\infty} \frac{(-aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, \tilde{C}^{(a)}_{2k+1}(\sqrt{a}\, q^{s+1};q)\, \tilde{C}^{(a)}_{2k'+1}(\sqrt{a}\, q^{s+1};q)$$

$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1+aq)\, a^{2k+1}}{(1+aq^{4k+3})} \frac{(q;q)_{2k+1}}{(-aq;q)_{2k+1}}\, q^{(k+2)(2k+1)}\, \delta_{kk'}$$

for the set of polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$. We see from this relation that *for $a>0$ the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, are orthogonal on the same set of points $\sqrt{a}\, q^{s+1}$, $s=0,1,2,\cdots$.*

Thus, the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, as well as the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, are orthogonal on the set of points $\sqrt{a}\, q^{s+1}$, $s=0,1,2,\cdots$. However, the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, are not orthogonal to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, on this set of points. As in the previous case, in order to prove that the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, are orthogonal to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$, one has to consider them on the set of points $\pm\sqrt{a}\, q^{s+1}$, $s=0,1,2,\cdots$. Since the polynomials from the first set are even and the polynomials from the second set are odd, then the infinite sum

$$I_1 \equiv \sum_{s=0}^{\infty} \frac{(-aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, \tilde{C}^{(a)}_{2k}(\sqrt{a}\, q^{s+1};q)\, \tilde{C}^{(a)}_{2k'+1}(\sqrt{a}\, q^{s+1};q)$$

coincides with the sum

$$I_2 \equiv -\sum_{s=0}^{\infty} \frac{(-aq^2;q^2)_s\, q^s}{(q^2;q^2)_s}\, \tilde{C}^{(a)}_{2k}(-\sqrt{a}\, q^{s+1};q)\, \tilde{C}^{(a)}_{2k'+1}(-\sqrt{a}\, q^{s+1};q).$$

Consequently, $I_1-I_2=0$. This gives the mutual orthogonality of the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k=0,1,2,\cdots$, to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k=0,1,2,\cdots$. Thus, the orthogonality

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relation for the whole set of polynomials $\tilde{C}_n^{(a)}(x;q)$, $n=0,1,2,\cdots$, can be written in the form

$$\sum_{s=0}^{\infty}\sum_{\varepsilon=\pm1}\frac{(-aq^2;q^2)_s\,q^s}{(q^2;q^2)_s}\,\tilde{C}_n^{(a)}(\varepsilon\sqrt{a}\,q^{s+1};q)\,\tilde{C}_{n'}^{(a)}(\varepsilon\sqrt{a}\,q^{s+1};q)$$

$$=\frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty}\,\frac{(1+aq)\,a^n}{(1+aq^{2n+1})}\,\frac{(q;q)_n}{(-aq;q)_n}\,q^{n(n+3)/2}\,\delta_{nn'}\,. \tag{5.14}$$

Note that the family of polynomials $\tilde{C}_n^{(a)}(x;q)$, $n=0,1,2,\cdots$, corresponds to the determinate moment problem, since the set of orthogonality is bounded. Thus, the orthogonality measure in (5.14) is unique.

In fact, formula (5.14) extends the orthogonality relation for the big q-Jacobi polynomials $P_n(x;a,a,-a;q)$ to a new domain of values of the parameter a.

5.3. Dual discrete q-ultraspherical polynomials. The polynomials (4.23) are dual to the big q-Jacobi polynomials (4.2). Let us set $a=b=-c$ in the polynomials (4.23), as we made before in the polynomials (4.2). This gives the polynomials

$$D_n^{(a^2)}(\mu(x;a^2)|q):=D_n(\mu(x;a^2);a,a,-a|q)$$

$$:=\ {}_3\phi_2\left(\begin{matrix}q^{-x},a^2q^{x+1},q^{-n}\\ aq,-aq\end{matrix}\,\middle|\,q,-q^{n+1}\right), \tag{5.15}$$

where $\mu(x;a^2)=q^{-x}+a^2q^{x+1}$. They satisfy the three-term recurrence relation

$$(q^{-x}+aq^{x+1})\,D_n^{(a)}(\mu(x;a)|q)=-q^{-2n-1}\,(1-aq^{2n+2})\,D_{n+1}^{(a)}(\mu(x;a)|q)$$

$$+\,q^{-2n-1}\,(1+q)\,D_n^{(a)}(\mu(x;a)|q)-\ q^{-2n}\,(1-q^{2n})\,D_{n-1}^{(a)}(\mu(x;a)|q),$$

which follows from the recurrence relation for the polynomials $D_n(\mu(x;ab);a,b,c|q)$ from section 4.3.

For the polynomials $D_n^{(a^2)}(\mu(x;a^2)|q)$ with imaginary a we introduce the notation

$$\tilde{D}_n^{(a^2)}(\mu(x;-a^2)|q):=D_n(\mu(x;-a^2);\mathrm{i}a,\mathrm{i}a,-\mathrm{i}a|q)$$

$$:=\ {}_3\phi_2\left(\begin{matrix}q^{-x},-a^2q^{x+1},q^{-n}\\ \mathrm{i}aq,-\mathrm{i}aq\end{matrix}\,\middle|\,q,-q^{n+1}\right)\,. \tag{5.16}$$

The polynomials $\tilde{D}_n^{(a)}(\mu(x;-a^2)|q)$ satisfy the recurrence relation

$$(q^{-x}-aq^{x+1})\,\tilde{D}_n^{(a)}(\mu(x;-a)|q)=-\,q^{-2n-1}\,(1+aq^{2n+2})\,\tilde{D}_{n+1}^{(a)}(\mu(x;-a)|q)$$

$$+\,q^{-2n-1}\,(1+q)\,\tilde{D}_n^{(a)}(\mu(x;-a)|q)-\ q^{-2n}\,(1-q^{2n})\,\tilde{D}_{n-1}^{(a)}(\mu(x;-a)|q).$$

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It is obvious from this relation that the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ are real for $x \in \mathbb{R}$ and $a > 0$. For $a > 0$ these polynomials satisfy the conditions of Favard's theorem and, therefore, they are orthogonal.

PROPOSITION 5.2. *The following expressions for the dual discrete q-ultraspherical polynomials* (5.15) *hold:*

$$D_n^{(a)}(\mu(2k;a)|q) = d_n(\mu(k;q^{-1}a);q^{-1},a|q^2)$$

$$= {}_3\phi_1\left(\begin{matrix} q^{-2k},\ aq^{2k+1},\ q^{-2n} \\ aq^2 \end{matrix}\,\middle|\, q^2, q^{2n+1}\right), \tag{5.17}$$

$$D_n^{(a)}(\mu(2k+1;a)|q) = q^n d_n(\mu(k;qa);q,a|q^2)$$

$$= q^n\, {}_3\phi_1\left(\begin{matrix} q^{-2k},\ aq^{2k+3},\ q^{-2n} \\ aq^2 \end{matrix}\,\middle|\, q^2, q^{2n-1}\right), \tag{5.18}$$

where k are nonnegative integers and $d_n(\mu(x;bc);b,c|q)$ are the dual little q-Jacobi polynomials (3.17).

Proof. Applying to the right side of (5.15) the formula (III.13) from Appendix III in [21] and then Singh's quadratic relation (5.7) for terminating ${}_3\phi_2$ series, after some transformations one obtains

$$D_n^{(a^2)}(\mu(2k;a^2)|q) = a^{-2k}q^{-k(2k+1)}\,{}_3\phi_2\left(\begin{matrix} q^{-2k},\ a^2q^{2k+1},\ a^2q^{2n+2} \\ a^2q^2,\ 0 \end{matrix}\,\middle|\, q^2, q^2\right).$$

Now apply the relation (0.6.26) from [27] in order to get

$$D_n^{(a^2)}(\mu(2k;a^2)|q) = \frac{(q^{-2k+1};q^2)_k}{(a^2q^2;q^2)_k}\,{}_2\phi_1\left(\begin{matrix} q^{-2k},\ a^2q^{2k+1} \\ q \end{matrix}\,\middle|\, q^2, q^{2n+2}\right).$$

Using formula (III.8) of Appendix III from [21], one arrives at the expression for the polynomials $D_n^{(a^2)}(\mu(2k;a^2)|q)$ in terms of the basic hypergeometric function from (5.17), coinciding with $d_n(\mu(k;q^{-1}a^2);q^{-1},a^2|q^2)$.

The formula (5.18) is proved in the same way by using the relation (5.11). Proposition is proved. $\square$

For the polynomials $\tilde{D}_n^{(a)}(\mu(m;-a)|q)$ with nonnegative integers m, we have the expressions

$$\tilde{D}_n^{(a)}(\mu(2k;-a)|q) = d_n(\mu(k;-q^{-1}a);q^{-1},-a|q^2)$$

$$= {}_3\phi_1\left(\begin{matrix} q^{-2k},\ -aq^{2k+1},\ q^{-2n} \\ -aq^2 \end{matrix}\,\middle|\, q^2, q^{2n+1}\right), \tag{5.19}$$

$$\tilde{D}_n^{(a)}(\mu(2k{+}1;-a)|q) = q^n d_n(\mu(k;-qa);q,-a|q^2)$$

$$= q^n\,{}_3\phi_1\left(\begin{matrix} q^{-2k}, -aq^{2k+3}, q^{-2n} \\ -aq^2 \end{matrix}\,\middle|\, q^2, q^{2n-1}\right). \tag{5.20}$$

It is plain from the explicit formulas that the polynomials $D_n^{(a)}(\mu(m)|q)$ and $\tilde{D}_n^{(a)}(\mu(m)|q)$ are rational functions of a.

5.4. Orthogonality relations for dual discrete q-ultraspherical polynomials. An example of the orthogonality relation for

$$D_n^{(a^2)}(\mu(x;a^2)|q) \equiv D_n(\mu(x;a^2);a,a,-a|q), \quad 0<a<q^{-1},$$

has been discussed in section 4. However, these polynomials correspond to the indeterminate moment problem and, therefore, this orthogonality relation is not unique. Let us find other orthogonality relations. In order to derive them we take into account the relations (5.17) and (5.18), and the orthogonality relation (3.18) for the dual little q-Jacobi polynomials. By means of formula (5.17), we arrive at the following orthogonality relation for $0<a<q^{-2}$:

$$\sum_{k=0}^{\infty} \frac{(1-aq^{4k+1})(aq;q)_{2k}}{(1-aq)(q;q)_{2k}}\, q^{k(2k-1)}\, D_n^{(a)}(\mu(2k)|q)\, D_{n'}^{(a)}(\mu(2k)|q)$$

$$= \frac{(aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(q^2;q^2)_n\, q^{-n}}{(aq^2;q^2)_n}\, \delta_{nn'},$$

where $\mu(2k) \equiv \mu(2k;a)$. The relation (5.18) leads to the orthogonality, which can be written in the form

$$\sum_{k=0}^{\infty} \frac{(1-aq^{4k+3})(aq;q)_{2k+1}}{(1-aq)(q;q)_{2k+1}}\, q^{k(2k+1)}\, D_n^{(a)}(\mu(2k+1)|q)\, D_{n'}^{(a)}(\mu(2k+1)|q)$$

$$= \frac{(aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(q^2;q^2)_n\, q^{-n}}{(aq^2;q^2)_n}\delta_{nn'},$$

where $\mu(2k+1) \equiv \mu(2k+1;a)$ and $0<a<q^{-2}$.

Thus, we have obtained two orthogonality relations for the polynomials $D_n^{(a)}(\mu(x;a)|q)$, $0<a<q^{-2}$, one on the lattice $\mu(2k;a) \equiv q^{-2k}+aq^{2k+1}$, $k=0,1,2,\cdots$, and another on the lattice $\mu(2k+1;a) \equiv q^{-2k-1}+aq^{2k+3}$, $k=0,1,2,\cdots$. *The corresponding orthogonality measures are extremal* since they are extremal for the dual little q-Jacobi polynomials from formulas (5.17) and (5.18) (see section 3).

The polynomials $\tilde{D}_n^{(a)}(\mu(x;a)|q)$ also correspond to the indeterminate moment problem and, therefore, they have infinitely many positive orthogonality measures. Some of their orthogonality relations can be derived in the same manner as for the polynomials $D_n^{(a)}(\mu(x)|q)$ by using the connection (5.19) and (5.20) of these polynomials with the dual little q-Jacobi polynomials (3.17). The relation (5.19) leads to the orthogonality relation

$$\sum_{k=0}^{\infty} \frac{(1+aq^{4k+1})(-aq;q)_{2k}}{(1+aq)(q;q)_{2k}}\, q^{k(2k-1)}\, \tilde{D}_n^{(a)}(\mu(2k)|q)\, \tilde{D}_{n'}^{(a)}(\mu(2k)|q)$$

$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(q^2;q^2)_n\, q^{-n}}{(-aq^2;q^2)_n}\, \delta_{nn'},$$

where $\mu(2k) \equiv \mu(2k;-a)$, and the relation (5.20) gives rise to the orthogonality relation, which can be written in the form

$$\sum_{k=0}^{\infty} \frac{(1+aq^{4k+3})(-aq;q)_{2k+1}}{(1+aq)(q;q)_{2k+1}}\, q^{k(2k+1)}\, \tilde{D}_n^{(a)}(\mu(2k+1)|q)\, \tilde{D}_{n'}^{(a)}(\mu(2k+1)|q)$$

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$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(q^2;q^2)_n q^{-n}}{(-aq^2;q^2)_n} \delta_{nn'},$$

where $\mu(2k+1) \equiv \mu(2k+1;-a)$. In both cases, a is any positive number.

Thus, in the case of the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ we also have two orthogonality relations. *The corresponding orthogonality measures are extremal* since they are extremal for the dual little q-Jacobi polynomials from formulas (5.19) and (5.20).

Note that the extremal measures for the polynomials $D_n^{(a)}(\mu(x)|q)$ and $\tilde{D}_n^{(a)}(\mu(x)|q)$, discussed in this section, can be used for constructing self-adjoint extensions of the closed symmetric operators, connected with the three-term recurrence relations for these polynomials and representable in an appropriate basis by a Jacobi matrix (details of such construction are given in [15], Chapter VII). These operators are representation operators for discrete series representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$ (see, for example, [26] for a description of this algebra). Moreover, the parameter a for these polynomials is connected with the number l, which characterizes the corresponding representation T_l of the discrete series.

5.5. Other orthogonality relations. The polynomials $D_n^{(a)}(\mu(x;a)|q)$ and the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ correspond to the indeterminate moment problems. For this reason, there exist infinitely many orthogonality relations for them. Let us derive some set of these relations for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, by using orthogonality relations for the polynomials (5.18) in [17]. These polynomials are (up to a factor) of the form

$$u_n((e^\xi - e^{-\xi})/2; t_1, t_2|q) = {}_3\phi_1\left(\begin{matrix} q\,e^\xi/t_1,\ -q\,e^{-\xi}/t_1,\ q^{-n} \\ -q^2/t_1t_2 \end{matrix} \middle| q, q^n t_1/t_2 \right) \tag{5.21}$$

and orthogonality relations, parameterized by a number d, $q \le d < 1$, are given by the formula

$$\sum_{n=-\infty}^{\infty} \frac{(-t_1q^{-n}/d, t_1q^n d, -t_2q^{-n}/d, t_2q^n d; q)_\infty}{(-t_1t_2/q;q)_\infty} \frac{d^{4n}q^{n(2n-1)}(1+d^2q^{2n})}{(-d^2;q)_\infty(-q/d^2;q)_\infty(q;q)_\infty}$$

$$\times u_r\left((d^{-1}q^{-n} - dq^n)/2; t_1, t_2\right) u_s\left((d^{-1}q^{-n} - dq^n)/2; t_1, t_2\right) = \frac{(q;q)_r (t_1/t_2)^r}{(-q^2/t_1t_2;q)_r\, q^r} \delta_{rs}. \tag{5.22}$$

The orthogonality measure here is positive for $t_1, t_2 \in \mathbb{R}$ and $t_1t_2 > 0$. It is not known whether these measures are extremal or not.

In order to use this orthogonality relation for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, let us consider the transformation formula

$${}_3\phi_2\left(\begin{matrix} q^{-2k}, -a^2q^{2k+1}, q^{-n} \\ \mathrm{i}aq, -\mathrm{i}aq \end{matrix} \middle| q, -q^{n+1} \right)$$

$$= {}_3\phi_1\left(\begin{matrix} q^{-2k}, -a^2q^{2k+1}, q^{-2n} \\ -a^2q^2 \end{matrix} \middle| q^2, q^{2n+1} \right), \tag{5.23}$$

which is true for any nonnegative integer values of k. It is obtained by equating two expressions (5.16) and (5.19) for the dual discrete q-ultraspherical polynomials $\tilde{D}_n^{(a)}(\mu(2k;-a)|q)$. Observe that (5.23) is still valid if one replaces numerator parameters q^{-2k} and $-a^2q^{2k+1}$

in both sides of it by $c^{-1}q^{-2k}$ and $-ca^2\,q^{2k+1}$, $c\in\mathbb{C}$, respectively. Indeed, the left side of (5.23) represents a finite sum:

$$ {}_3\phi_2(q^{-n},\alpha,\beta;\ \gamma,\delta;\ q,z) := \sum_{m=0}^{n}\frac{(q^{-n},\alpha,\beta;q)_m}{(\gamma,\delta,q;q)_m}z^m. \tag{5.24}$$

In the case in question $\alpha = q^{-2k}$ and $\beta = -a^2q^{2k+1}$, so the q-shifted factorial $(\alpha,\beta;q)_m$ in (5.24) is equal to

$$(q^{-2k},-a^2q^{2k+1};q)_m$$

$$=\prod_{j=0}^{m-1}[1-a^2q^{2j+1}-q^j(q^{-2k}-a^2q^{2k+1})]=\prod_{j=0}^{m-1}[1-a^2q^{2j+1}-q^j\mu(2k;-a^2)],$$

where, as before, $\mu(2k;-a^2) = q^{-2k} - a^2q^{2k+1}$. The left side in (5.23) thus represents a polynomial $p_n(x)$ in the $\mu(2k;-a^2)$ of degree n. In a similar manner, one easily verifies that the right side of (5.23) also represents a polynomial $p'_n(x)$ in the same $\mu(2k;-a^2)$ of degree n. In other words, the transformation formula (5.23) states that the polynomials $p_n(x)$ and $p'_n(x)$ are equal to each other on the infinite set of distinct points $x_k = \mu(2k;-a^2)$. Thus, they are identical.

An immediate consequence of this statement is that (5.23) still holds if one replaces the numerator parameters q^{-2k} and a^2q^{2k+1} in both sides of (5.23) by $c^{-1}q^{-2k}$ and ca^2q^{2k+1}, respectively. The point is that

$$(c^{-1}q^{-2k},-ca^2q^{2k+1};q)_m=\prod_{j=0}^{m-1}[1-a^2q^{2j+1}-q^j(c^{-1}q^{-2k}-ca^2q^{2k+1})]$$

$$=\prod_{j=0}^{m-1}[1-a^2q^{2j+1}-q^j\mu_c(2k;-a^2)],$$

where $\mu_c(2k;-a^2) = c^{-1}q^{-2k} - ca^2q^{2k+1}$. So, the replacements $q^{-2k}\to c^{-1}q^{-2k}$ and $a^2q^{2k+1}\to ca^2q^{2k+1}$ change only the variable: $\mu(2k;-a^2)\to\mu_c(2k;-a^2)$, whereas all other parameters in both sides of (5.23) are unaltered. Thus, our statement is proved.

We are now in a position to establish other orthogonality relations for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, distinct from those, obtained in subsection 5.4. To achieve this, we use the fact that the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ at the points $x = x_k^{(d)} := 2k - \ln(\sqrt{aq}/d)/\ln q$ are equal to

$$\tilde{D}_n^{(a)}(\mu(x_k^{(d)};-a)|q) = {}_3\phi_2\left(\begin{matrix} q^{-2k}d^{-1}\sqrt{aq},\ -q^{2k}d\sqrt{aq},\ q^{-n} \\ \mathrm{i}\sqrt{a}\,q,\ -\mathrm{i}\sqrt{a}\,q\end{matrix}\,\middle|\, q,-q^{n+1}\right),$$

where $\mu(x_k^{(d)};-a) = \sqrt{aq}\,(d^{-1}\,q^{-2k} - d\,q^{2k})$. From (5.21) and (5.23) (with q^{-2k} and $-a^2q^{2k+1}$ replaced by $d^{-1}q^{-2k}$ and $-da^2q^{2k+1}$, respectively) it then follows that

$$\tilde{D}_n^{(a)}(\mu(x_k^{(d)};-a)|q) = u_n\left((d^{-1}\,q^{-2k} - d\,q^{2k})/2;\sqrt{q^3/a},\sqrt{q/a}\,|q^2\right).$$

Hence, from the orthogonality relations (5.22) one obtains infinite number of orthogonality relations for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, which are parameterized by the same d as in (5.22). They are of the form

$$\sum_{n=-\infty}^{\infty} \frac{(-t_1 q^{-2n}/d, t_1 q^{2n} d, -t_2 q^{-2n}/d, t_2 q^{2n} d; q^2)_\infty}{(-t_1 t_2/q^2; q^2)_\infty} \frac{d^{4n} q^{2n(2n-1)}(1+d^2 q^{4n})}{(-d^2;q^2)_\infty (-q^2/d^2;q^2)_\infty (q^2;q^2)_\infty}$$

$$\times \tilde{D}_r^{(a)}(\mu(x_n^{(d)};-a)|q)\, \tilde{D}_s^{(a)}(\mu(x_n^{(d)};-a)|q) = \frac{(q^2;q^2)_r}{(-q^2 a;q^2)_r^2}\, \delta_{rs}, \tag{5.25}$$

where $t_1 = \sqrt{q^3/a}$ and $t_2 = \sqrt{q/a}$.

There exist yet another connection of the polynomials $\tilde{D}_n^{(a)}(\mu(x)|q)$ with the polynomials (5.21). In order to obtain it we consider the relation

$${}_3\phi_2\left(\begin{matrix} q^{-2k-1}, -a^2 q^{2k+2}, q^{-n} \\ \mathrm{i}aq, -\mathrm{i}aq \end{matrix} \,\middle|\, q, -q^{n+1}\right)$$

$$= q^n\, {}_3\phi_1\left(\begin{matrix} q^{-2k}, -a^2 q^{2k+3}, q^{-2n} \\ -a^2 q^2 \end{matrix} \,\middle|\, q^2, q^{2n-1}\right). \tag{5.26}$$

This relation is true for nonnegative integer values of k. However, it can be proved (in the same way as in the case of formula (5.23)) that it holds also if we replace q^{2k} and q^{-2k} by cq^{2k} and $c^{-1}q^{-2k}$, respectively.

As in the previous case, we put $x = x_k^{(d)} := 2k - \ln(\sqrt{aq}/d)/\ln q$, that is, $\mu(x_k^{(d)};-a) = \sqrt{aq}\left(d^{-1}\, q^{-2k} - d\, q^{2k}\right)$. Then by means of formula (5.26) (with q^{-2k} and q^{2k} replaced by $d^{-1}q^{-2k}$ and dq^{2k}, respectively), we derive that

$$\tilde{D}_n^{(a)}(\mu(x_k^{(d)};-a)|q) = q^n\, u_n((d^{-1}q^{-2k} - dq^{2k})/2; \sqrt{q/a}, \sqrt{q^3/a}|q^2).$$

We may apply to the polynomials $u_n(x;\sqrt{q/a},\sqrt{q^3/a}|q^2)$ the orthogonality relations (5.22) and obtain an infinite number of orthogonality relations for the polynomials $\tilde{D}_n^{(a)}(\mu(y)|q)$. However, they coincide with the orthogonality relations (5.25).

It is important to know whether an orthogonality measure for polynomials is extremal or not. The extremality of the measures in (5.25) for the polynomials $\tilde{D}_n^{(a)}(\mu_c(x;-a)|q)$ depends on the extremality of the orthogonality measures in (5.22) for the polynomials (5.21). If some of the measures in (5.22) are extremal, then the corresponding measures in (5.25) are also extremal.

6. Duality of big q-Laguerre and q-Meixner polynomials.

6.1. Operators related to big q-Laguerre polynomials. Let $\mathcal{H} \equiv \mathcal{H}_a$ be the Hilbert space of functions, introduced in subsection 3.1, which is fixed by a real number a, such that $0 < a < q^{-1}$. In this section we are interested in the operator

$$A\, f_n = r_{n+1} f_{n+1} + r_n f_{n-1} + d_n f_n, \tag{6.1}$$

where

$$r_n = (-abq^{n+2})^{1/2} \sqrt{(1-q^n)(1-aq^n)(1-bq^n)},$$
$$d_n = -abq^{2n+1}(1+q) + q^{n+1}(a+ab+b),$$

a is the same as above and b is a fixed negative number.

Since $q < 1$ the operator A is bounded. Therefore, one can close this operator and we assume in what follows that A is a closed (and consequently defined on the whole space $\mathcal{H}$) operator. Since A is symmetric, its closure is a self-adjoint operator. In the same way as in subsection 3.1, one readily proves that A is a Hilbert–Schmidt operator. Therefore, A has the discrete spectrum.

We wish to find eigenfunctions $\xi_\lambda(x)$ of the operator A, $A\xi_\lambda(x) = \lambda\xi_\lambda(x)$. We set

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} a_n(\lambda) f_n(x).$$

Acting by A upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} a_n(\lambda)[r_{n+1}\, f_{n+1} + r_n\, f_{n-1} + d_n\, f_n] = \lambda \sum_{n=0}^{\infty} a_n(\lambda) f_n.$$

Comparing coefficients of a fixed f_n, one obtains a three-term recurrence relation for the $a_n(\lambda)$:

$$r_{n+1}\, a_{n+1}(\lambda) + r_n\, a_{n-1}(\lambda) + d_n\, a_n(\lambda) = \lambda\, a_n(\lambda).$$

Making the substitution

$$a_n(\lambda) = (-ab)^{-n/2}\, q^{-n(n+3)/4} \left(\frac{(aq, bq; q)_n}{(q;q)_n}\right)^{1/2} a'_n(\lambda)$$

and using the explicit expressions for the r_n and d_n, we derive the relation

$$(1 - aq^{n+1})(1 - bq^{n+1})\, a'_{n+1}(\lambda) - ab\, q^{n+1}\,(1 - q^n)\, a'_{n-1}(\lambda) + d_n\, a'_n(\lambda) = \lambda\, a'_n(\lambda),$$

where, as before, $0 < a < q^{-1}$ and $b < 0$. It coincides with the recurrence relation for the big q-Laguerre polynomials, which are defined as

$$P_n(\lambda; a, b; q) := {}_3\phi_2(q^{-n}, 0, \lambda;\; aq, bq;\; q, q)$$

$$= (q^{-n}/b; q)_n^{-1}\; {}_2\phi_1(q^{-n},\, aq/\lambda;\; aq;\; q, \lambda/b) \tag{6.2}$$

(see formula (3.11.3) in [27]), that is, $a'_n(\lambda) = P_n(\lambda; a, b; q)$. Therefore,

$$a_n(\lambda) = (-ab)^{-n/2}\, q^{-n(n+3)/4} \left(\frac{(aq, bq; q)_n}{(q;q)_n}\right)^{1/2} P_n(\lambda; a, b; q). \tag{6.3}$$

Thus, the eigenfunctions of the operator A have the form

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} (-ab)^{-n/2}\, q^{-n(n+3)/4} \left(\frac{(aq, bq; q)_n}{(q;q)_n}\right)^{1/2} P_n(\lambda; a, b; q)\, f_n(x)$$

$$= \sum_{n=0}^{\infty} (-b)^{-n/2}\, a^{-3n/4}\, q^{-n(n+3)/4}\, \frac{(aq;q)_n}{(q;q)_n} (bq;q)_n^{1/2}\, P_n(\lambda; a, b; q)\, x^n. \tag{6.4}$$

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Since the spectrum of the operator A is discrete, only a countable set of these functions belongs to the Hilbert space $\mathcal{H}$. This discrete set of functions determines a spectrum of A.

In order to be able to find a spectrum of the operator A, we consider the linear operator q^{-J_0} which is diagonal in the basis $\{f_n\}$ and is given by the formula

$$q^{-J_0} f_n = (aq)^{-1/2} q^{-n} f_n$$

(see formula (3.1)). We have to find how the operator q^{-J_0} acts upon the eigenfunctions $\xi_\lambda(x)$ of the operator A (which belong to the Hilbert space $\mathcal{H}$). In order to do this one can use the q-difference equation

$$q^{-n}(1-q^n)\lambda^2 p_n(\lambda) = B(\lambda) p_n(q\lambda) - [B(\lambda)+D(\lambda)] p_n(\lambda) + D(\lambda) p_n(q^{-1}\lambda) \tag{6.5}$$

for the big q-Laguerre polynomials $p_n(\lambda) \equiv P_n(\lambda; a, b; q)$, where

$$B(\lambda) = abq\,(1-\lambda), \quad D(\lambda) = (\lambda - aq)\,(\lambda - bq).$$

Multiply both sides of (6.5) by $k_n f_n(x)$ and sum up over n, where k_n are the coefficients of $P_n(\lambda; a, b; q)$ in the expression (6.3) for the coefficients $a_n(\lambda)$. Taking into account formula (6.4) and the form of the operator q^{-J_0} in the basis $\{f_n\}$, one obtains the relation

$$(aq)^{1/2} q^{-J_0} \lambda^2 \xi_\lambda(x) = B(\lambda)\xi_{q\lambda}(x) - [B(\lambda)+D(\lambda)-\lambda^2]\,\xi_\lambda(x) + D(\lambda)\,\xi_{q^{-1}\lambda}(x), \tag{6.6}$$

where $B(\lambda)$ and $D(\lambda)$ are the same as in (6.5).

6.2. Spectrum of A and orthogonality of big q-Laguerre polynomials. The aim of this subsection is to find a basis in the Hilbert space $\mathcal{H}$, which consists of eigenfunctions of the operator A in a normalized form, and to derive explicitly the unitary matrix U, connecting this basis with the canonical basis f_n, $n = 0, 1, 2, \cdots$, in $\mathcal{H}$. This matrix directly leads to the orthogonality relation for the big q-Laguerre polynomials.

Exactly as in section 4, one can show that for some value of λ (which belongs to the spectrum) the last term on the right side of (6.6) has to vanish. There are two such values of λ: $\lambda = aq$ and $\lambda = bq$, which are the roots of the equation $D(\lambda) = 0$. Let us show that both of these points are spectral points of the operator A. Due to (6.2) we have

$$P_n(aq; a, b; q) = (q^{-n}/b; q)_n^{-1} = \frac{(-bq)^n q^{n(n-1)/2}}{(bq; q)_n},$$

$$P_n(bq; a, b; q) = \frac{(-aq)^n q^{n(n-1)/2}}{(aq; q)_n}.$$

Hence, for the scalar product $\langle \xi_{aq}(x), \xi_{aq}(x)\rangle$ we have the expression

$$\sum_{n=0}^{\infty} \frac{(aq, bq; q)_n}{(-ab)^n q^{n(n+3)/2}\,(q; q)_n} P_n^2(aq; a, b; q) = \sum_{n=0}^{\infty} (-b/a)^n\, q^{n(n-1)/2} \frac{(aq; q)_n}{(bq; q)_n (q; q)_n}$$

$$= {}_1\phi_1(a; b; q, b/a) = \frac{(b/a; q)_\infty}{(b; q)_\infty} < \infty. \tag{6.7}$$

Similarly, for $\langle \xi_{bq}(x), \xi_{bq}(x)\rangle$ one has the expression

$$\sum_{n=0}^{\infty} \frac{(aq, bq; q)_n}{(-ab)^n\, q^{n(n+3)/2}\,(q; q)_n} P_n^2(bq; a, b; q) = \frac{(a/b; q)_\infty}{(a; q)_\infty} < \infty. \tag{6.8}$$

Thus, the values $\lambda = aq$ and $\lambda = bq$ are the spectral points of the operator A.

Let us find other spectral points of the operator A. Setting $\lambda = aq$ in (6.6), we see that the operator q^{-J_0} transforms $\xi_{aq}(x)$ into a linear combination of the functions $\xi_{aq^2}(x)$ and $\xi_{aq}(x)$. We have to show that $\xi_{aq^2}(x)$ belongs to the Hilbert space $\mathcal{H}$, that is, that

$$\langle \xi_{aq^2}, \xi_{aq^2} \rangle = \sum_{n=0}^{\infty} \frac{(aq, bq; q)_n}{(-ab)^n q^{n(n+3)/2}\, (q;q)_n}\, P_n^2(aq^2; a, b; q) \; < \; \infty.$$

In order to achieve this we note that since $(aq^2;q)_k = (aq;q)_k(1-aq^{k+1})/(1-aq)$, we have

$$P_n(aq^2; a, b; q) = \sum_{k=0}^{n} \frac{1-aq^{k+1}}{1-aq} \frac{(q^{-n};q)_k (aq;q)_k}{(aq;q)_k (bq;q)_k} \frac{q^k}{(q;q)_k}$$

$$\le \frac{1}{1-aq} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (aq;q)_k}{(aq;q)_k (bq;q)_k} \frac{q^k}{(q;q)_k} = (1-aq)^{-1} P_n(aq; a, b; q).$$

Therefore, the series for $\langle \xi_{aq^2}, \xi_{aq^2} \rangle$ is majorized (up to the finite constant $(1-aq)^{-1}$) by the corresponding series for $\langle \xi_{aq}, \xi_{aq} \rangle$. Thus, $\xi_{aq^2}(x)$ is an eigenfunction of A and the point aq^2 belongs to the spectrum of the operator A. Setting $\lambda = aq^2$ in (6.6) and acting similarly, one obtains that $\xi_{aq^3}(x)$ is an eigenfunction of A and the point aq^3 belongs to the spectrum of A. Repeating this procedure, one sees that $\xi_{aq^n}(x)$, $n = 1, 2, \cdots$, are eigenfunctions of A and the set aq^n, $n = 1, 2, \cdots$, belongs to the spectrum of A. Likewise, one concludes that $\xi_{bq^n}(x)$, $n = 1, 2, \cdots$, are eigenfunctions of A and the set bq^n, $n = 1, 2, \cdots$, belongs to the spectrum of A. So far we do not know whether the operator A has other spectral points or not. In order to solve this problem we shall proceed as in subsection 3.2.

The functions $\xi_{aq^n}(x)$ and $\xi_{bq^n}(x)$, $n = 1, 2, \cdots$, are linearly independent elements of the Hilbert space $\mathcal{H}$. Suppose that aq^n and bq^n, $n = 1, 2, \cdots$, constitute the whole spectrum of the operator A. Then the set of functions $\xi_{aq^n}(x)$ and $\xi_{bq^n}(x)$, $n = 1, 2, \cdots$, is a basis in the space $\mathcal{H}$. Introducing the notations $\Xi_n := \xi_{aq^{n+1}}(x)$ and $\Xi'_n := \xi_{bq^{n+1}}(x)$, $n = 0, 1, 2, \cdots$, and taking into account the relation $B(\lambda) + D(\lambda) - \lambda^2 = abq(1+q) - \lambda q(ab + a + b)$, we find from (6.6) that

$$q^{-J_0}\Xi_n = a^{-3/2} b q^{-2n-3/2}(1-aq^{n+1})\Xi_{n+1} - a^{-3/2} q^{-2n-3/2}[b(1+q) - q^{n+1}(ab+a+b)]\Xi_n$$

$$+ a^{-3/2} b q^{-2n-1/2}(1-q^n)(1-aq^n/b)\Xi_{n-1}$$

for $\lambda = aq^{n+1}$, that is, for $\xi_{aq^{n+1}}(x) = \Xi_n(x)$. Similarly,

$$q^{-J_0}\Xi'_n = a^{1/2} b^{-1} q^{-2n-3/2}(1-bq^{n+1})\Xi'_{n+1} - a^{1/2} b^{-1} q^{-2n-3/2}$$

$$\times [1 + q - a^{-1} q^{n+1}(ab + a + b)]\Xi'_n + a^{1/2} b^{-1} q^{-2n-1/2}(1-q^n)(1-bq^n/a)\Xi'_{n-1}$$

for $\lambda = bq^{n+1}$, that is, for $\xi_{bq^{n+1}}(x) = \Xi'_n(x)$.

As we see, the matrix of the operator q^{-J_0} in the basis $\Xi_n = \xi_{aq^{n+1}}(x)$, $\Xi'_n = \xi_{bq^{n+1}}(x)$, $n = 0, 1, 2, \cdots$, is not symmetric, although in the initial basis f_n, $n = 0, 1, 2, \cdots$, it was symmetric. The reason is that the matrix $M := ((b_{mn})_{m,n=0}^{\infty}\ (b'_{mn})_{m,n=0}^{\infty})$ with entries

$$b_{mn} := a_m(aq^n), \qquad b'_{mn} := a_m(bq^n), \qquad m, n = 0, 1, 2, \cdots,$$

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where $a_m(dq^n)$, $d = a, b$, are coefficients (6.3) in the expansion

$$\xi_{dq^n}(x) = \sum_m a_m(dq^n)\, f_n(x)$$

(see (6.4)), is not unitary. It maps the basis $\{f_n\}$ into the basis $\{\xi_{aq^{n+1}}, \xi_{bq^{n+1}}\}$ in the Hilbert space $\mathcal{H} \equiv \mathcal{H}_a$. The nonunitarity of the matrix M is equivalent to the statement that the basis $\Xi_n := \xi_{aq^{n+1}}(x)$, $\Xi'_n := \xi_{bq^{n+1}}(x)$, $n = 0, 1, 2, \cdots$, is not normalized. In order to normalize it, we have to multiply Ξ_n by appropriate numbers c_n and Ξ'_n by numbers c'_n. Let $\hat{\Xi}_n = c_n \Xi_n$, $\hat{\Xi}'_n = c'_n \Xi_n$, $n = 0, 1, 2, \cdots$, be a normalized basis. Then the operator q^{-J_0} is symmetric in this basis and has the form

$$\begin{aligned} q^{-J_0}\,\hat{\Xi}_n = c_{n+1}^{-1} c_n a^{-3/2} b q^{-2n-3/2}(1 - aq^{n+1})\,\hat{\Xi}_{n+1} \\ - a^{-3/2} q^{-2n-3/2}[b(1+q) - q^{n+1}(ab + a + b)]\,\hat{\Xi}_n \\ + c_{n-1}^{-1} c_n a^{-3/2} b q^{-2n-1/2}(1 - q^n)(1 - aq^n/b)\,\hat{\Xi}_{n-1}, \end{aligned}$$

$$q^{-J_0}\,\hat{\Xi}'_n = c'^{-1}_{n+1} c'_n a^{1/2} b^{-1} q^{-2n-3/2}(1 - bq^{n+1})\,\hat{\Xi}'_{n+1} - a^{1/2} b^{-1} q^{-2n-3/2}$$

$$\times\,[1 + q - a^{-1} q^{n+1}(ab + a + b)]\,\hat{\Xi}'_n + c'^{-1}_{n-1} c'_n a^{1/2} b^{-1} q^{-2n-1/2}(1 - q^n)(1 - bq^n/a)\,\hat{\Xi}'_{n-1}.$$

The symmetricity of the matrix of the operator q^{-J_0} in the basis $\{\hat{\Xi}_n, \hat{\Xi}'_n\}$ means that

$$c_{n+1}^{-1} c_n q^{-2n-3/2}(1 - aq^{n+1}) = c_n^{-1} c_{n+1} q^{-2n-5/2}(1 - q^{n+1})(1 - aq^{n+1}/b),$$

$$c'^{-1}_{n+1}\, c'_n\, q^{-2n-3/2}(1 - bq^{n+1}) = c'^{-1}_n\, c'_{n+1}\, q^{-2n-5/2}(1 - q^{n+1})(1 - bq^{n+1}/a),$$

that is,

$$\frac{c_n}{c_{n-1}} = \sqrt{q\frac{(1 - aq^n)}{(1 - q^n)(1 - aq^n/b)}}, \quad \frac{c'_n}{c'_{n-1}} = \sqrt{q\frac{(1 - bq^n)}{(1 - q^n)(1 - bq^n/a)}}.$$

Thus,

$$c_n = C\,\left(\frac{(aq;q)_n\, q^n}{(aq/b, q;q)_n}\right)^{1/2} \tag{6.9}$$

and

$$c'_n = C'\,\left(\frac{(bq;q)_n q^n}{(bq/a, q;q)_n}\right)^{\frac{1}{2}}, \tag{6.10}$$

where C and C' are some constants.

Therefore, in the expansions

$$\hat{\xi}_{aq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\, a_m(aq^n)\, f_m(x) = \sum_m \hat{b}_{mn}\, f_m(x), \tag{6.11}$$

$$\hat{\xi}'_{bq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c'_n\, a_m(bq^n)\, f_m(x) = \sum_m \hat{b}'_{mn}\, f_m(x), \tag{6.12}$$

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the matrix $\hat{M} := ((\hat{b}_{mn})_{m,n=0}^{\infty}\ (\hat{b}'_{mn})_{m,n=0}^{\infty})$ with entries

$$\hat{b}_{mn} = c_n\, a_m(aq^n) = C\, \left(\frac{(aq;q)_n q^n}{(aq/b,q;q)_n} \frac{(a,b;q)_m}{(q;q)_m\,(-ab)^m} \right)^{\frac{1}{2}}$$

$$\times\, q^{-m(m+3)/4}\, P_m(aq^{n+1};a,b;q), \tag{6.13}$$

$$\hat{b}'_{mn}{=}c'_n\, a_m(bq^n){=}C'\, \left(\frac{(bq;q)_n q^n}{((bq/a,q;q)_n} \frac{(a,b;q)_m}{(q;q)_m\,(-ab)^m} \right)^{\frac{1}{2}}$$

$$\times\, q^{-m(m+3)/4}\, P_m(bq^{n+1};a,b;q), \tag{6.14}$$

is unitary, provided that the constants C and C' are appropriately chosen. In order to calculate these constants, one can use the relations

$$\sum_{m=0}^{\infty} |\hat{b}_{mn}|^2 = 1\,, \quad \sum_{m=0}^{\infty} |\hat{b}'_{mn}|^2 = 1$$

for $n = 0$. Then these sums are multiples of the sums in (6.7) and (6.8), so we find that

$$C = \frac{(bq;q)_{\infty}^{1/2}}{(b/a;q)_{\infty}^{1/2}}, \qquad C' = \frac{(aq;q)_{\infty}^{1/2}}{(a/b;q)_{\infty}^{1/2}}.$$

The coefficients c_n and c'_n in (6.11)–(6.14) are thus real and equal to

$$c_n = \left(\frac{(aq;q)_n (bq;q)_{\infty}\, q^n}{(aq/b,q;q)_n (b/a;q)_{\infty}} \right)^{\frac{1}{2}} \tag{6.15}$$

and

$$c'_n = \left(\frac{(bq;q)_n (aq;q)_{\infty} q^n}{(bq/a,q;q)_n (a/b;q)_{\infty}} \right)^{\frac{1}{2}}. \tag{6.16}$$

The orthogonality of the matrix $\hat{M} \equiv (\hat{b}_{mn}\ \hat{b}'_{mn})$ means that

$$\sum_m \hat{b}_{mn}\hat{b}_{mn'} = \delta_{nn'}\,, \quad \sum_m \hat{b}'_{mn}\hat{b}'_{mn'} = \delta_{nn'}\,, \quad \sum_m \hat{b}_{mn}\hat{b}'_{mn'} = 0, \tag{6.17}$$

$$\sum_n (\hat{b}_{mn}\hat{b}_{m'n} + \hat{b}'_{mn}\hat{b}'_{m'n}) = \delta_{mm'}. \tag{6.18}$$

Substituting the expressions for $\hat{b}_{mn}$ and $\hat{b}'_{mn}$ into (6.18), one obtains the relation

$$\frac{(bq;q)_{\infty}}{(b/a;q)_{\infty}} \sum_{n=0}^{\infty} \frac{(aq;q)_n q^n}{(aq/b;q)_n (q;q)_n}\, P_m(aq^{n+1};a,b;q)\, P_{m'}(aq^{n+1};a,b;q)$$

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$$+\frac{(aq;q)_\infty}{(a/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(bq;q)_n q^n}{(bq/a;q)_n(q;q)_n}\,P_m(bq^{n+1};a,b;q)\,P_{m'}(bq^{n+1};a,b;q)$$

$$=\frac{(q;q)_m}{(aq,bq;q)_m}(-ab)^m q^{m(m+3)/2}\delta_{mm'}. \tag{6.19}$$

This identity must give an orthogonality relation for the big q-Laguerre polynomials $P_m(y) \equiv P_m(y;a,b;q)$. An only gap, which appears here, is the following. We have assumed that the points aq^n and bq^n, $n = 0,1,2,\cdots$, exhaust the whole spectrum of the operator A. As in the case of the operator I_2 in section 4, if the operator A had other spectral points x_k, then on the left-hand side of (6.20) would appear other summands $\mu_{x_k}\,P_m(x_k;a,b;q)\,P_{m'}(x_k;a,b;q)$, which correspond to these additional points. Let us show that these additional summands do not appear. To this end we set $m = m' = 0$ in the relation (6.19) with the additional summands. This results in the equality

$$\frac{(bq;q)_\infty}{(b/a;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n q^n}{(aq/b;q)_n(q;q)_n}+\frac{(aq;q)_\infty}{(a/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(bq;q)_n q^n}{(bq/a;q)_n(q;q)_n}+\sum_k \mu_{x_k}=1.$$

In terms of the ${}_2\phi_1$ basic hypergeometric series this identity can be written as

$$\frac{(bq;q)_\infty}{(b/a;q)_\infty}\,{}_2\phi_1(aq,0;\;aq/b;\;q,q)+\frac{(aq;q)_\infty}{(a/b;q)_\infty}{}_2\phi_1(bq,0;\;bq/a;\;q,q)+\sum_k \mu_{x_k}=1.$$

We recall that it represents a particular case of Sears' three-term transformation formula for ${}_2\phi_1(a,0;c;q,q)$ series (see [21], formula (3.3.5)) if $\mu_{x_k}=0$ for all values of k. Therefore, in (6.19) the sum $\sum_k \mu_{x_k}$ does really vanish and formula (6.19) gives an orthogonality relation for big q-Laguerre polynomials.

By using the operators A and q^{-J_0}, we thus derived the orthogonality relation for big q-Laguerre polynomials.

The orthogonality relation (6.19) for big q-Laguerre polynomials enables one to formulate the following statement: *The spectrum of the operator A coincides with the set of points aq^{n+1} and bq^{n+1}, $n = 0,1,2,\cdots$. The spectrum is simple and has one accumulation point at 0.*

6.3. Dual polynomials and functions. The matrix $M \equiv (\hat{b}_{mn}\;\hat{b}'_{mn})$ with entries (6.13) and (6.14) is unitary and it connects two orthonormal bases in the Hilbert space $\mathcal{H}$. The relations (6.17) for its matrix elements is the orthogonality relation for the functions, which are dual to the big q-Laguerre polynomials and are defined as

$$f_n(q^{-m};a,b|q) := P_m(aq^{n+1};a,b;q), \quad n=0,1,2,\cdots, \tag{6.20}$$

$$g_n(q^{-m};a,b|q) := P_m(bq^{n+1};a,b;q), \quad n=0,1,2,\cdots. \tag{6.21}$$

Taking into account the expressions for the entries $\hat{b}_{mn}$ and $\hat{b}'_{mn}$, the first two relations in (6.17) can be written as

$$\sum_{m=0}^{\infty} a_m(aq^{n+1})a_m(aq^{n'+1}) = c_n^{-2}\,\delta_{nn'}, \quad \sum_{m=0}^{\infty} a_m(bq^{n+1})a_m(bq^{n'+1}) = {c'}_n^{-2}\,\delta_{nn'}.$$

Substituting the explicit expressions for the coefficients $a_m(\lambda_n)$, we derive the following orthogonality relations for the functions (6.20) and (6.21):

$$\sum_{m=0}^{\infty} \frac{(aq, bq;q)_m}{(q;q)_m(-abq^2)^m}\, q^{-m(m-1)/2} f_n(q^{-m};a,b|q) f_{n'}(q^{-m};a,b|q) = c_n^{-2}\,\delta_{nn'}, \tag{6.22}$$

$$\sum_{m=0}^{\infty} \frac{(aq, bq;q)_m}{(q;q)_m(-abq^2)^m}\, q^{-m(m-1)/2}\, g_n(q^{-m};a,b|q)\, g_{n'}(q^{-m};a,b|q) = {c'}_n^{-2}\,\delta_{nn'}, \tag{6.23}$$

$$\sum_{m=0}^{\infty} \frac{(aq, bq;q)_m}{(q;q)_m(-abq^2)^m}\, q^{-m(m-1)/2}\, f_n(q^{-m};a,b|q)\, g_{n'}(q^{-m};a,b|q) = 0, \tag{6.24}$$

where c_n and c'_n are given by the formulas (6.15) and (6.16).

Comparing the expression

$$M_n(q^{-x};a,b;q) := {}_2\phi_1(q^{-n}, q^{-x};\ aq;\ q, -q^{n+1}/b)$$

for the q-Meixner polynomials with the explicit form (6.2) of the big q-Laguerre polynomials $P_m(x;a,b;q)$, we see that

$$f_n(q^{-m};a,b|q) = (q^{-m}/b;q)_m^{-1} M_n(q^{-m};a,-b/a;q).$$

Since $(q^{-m}/b;q)_m = (bq;q)_m(-b)^{-m}q^{-m(m+1)/2}$, the orthogonality relation (6.22) leads to the orthogonality relation for the q-Meixner polynomials $M_n(q^{-m}) \equiv M_n(q^{-m};a,-b/a;q)$:

$$\sum_{m=0}^{\infty} \frac{(aq;q)_m(-b/a)^m\, q^{m(m-1)/2}}{(bq,q;q)_m} M_n(q^{-m})\, M_{n'}(q^{-m})$$

$$= \frac{(b/a;q)_\infty}{(bq;q)_\infty} \frac{(aq/b,q;q)_n}{(aq;q)_n}\, q^{-n}\,\delta_{nn'}, \tag{6.25}$$

where, as before, $0 < a < q^{-1}$ and $b < 0$. This orthogonality relation coincides with the known formula for the q-Meixner polynomials (see, for example, (3.13.2) in [27]).

The functions (6.21) are also expressed in terms of q-Meixner polynomials. Indeed, we have

$$g_n(q^{-m};a,b|q) = {}_3\phi_2(q^{-m}, 0, bq^{n+1};\ aq, bq;\ q,q)$$

$$= (q^{-m}/a;q)_m^{-1}\, {}_2\phi_1(q^{-n}, q^{-m};\ bq;\ q, bq^{n+1}/a) = (q^{-m}/a;q)_m^{-1}\, M_n(q^{-m};b,-a/b;\ q),$$

where $b < 0$, that is, one of the parameters in these q-Meixner polynomials is negative.

Substituting this expression for $g_n(q^{-m};a,b|q)$ into (6.23), we obtain the orthogonality relation for q-Meixner polynomials $M_n(q^{-m}) \equiv M_n(q^{-m};b,-a/b;\ q)$ with negative b:

$$\sum_{m=0}^{\infty} \frac{(bq;q)_m(-a/b)^m}{(aq,q;q)_m}\, q^{m(m-1)/2}\, M_n(q^{-m})\, M_{n'}(q^{-m})$$

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$$= \frac{(a/b;q)_\infty}{(aq;q)_\infty} \frac{(bq/a,q;q)_n}{(bq;q)_n} q^{-n} \delta_{nn'}. \tag{6.26}$$

Observe that this orthogonality relation is of the same form as for $b > 0$. As far as we know, this type of orthogonality relation for negative values of the parameter b has been first discussed in [2].

The relation (6.24) can be written as the equality

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m-1)/2}}{(q;q)_m} M_n(q^{-m};a,-b/a;\ q)\, M_{n'}(q^{-m};b,-a/b;\ q) = 0,$$

which holds for $n, n' = 0, 1, 2, \cdots$. The validity of this identity for arbitrary nonnegative integers n and n' can be verified directly by using Jackson's q-exponential function

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty$$

and the fact that $E_q(z)$ has zeroes at the points $z_j = -q^{-j}$, $j = 0, 1, 2, \cdots$.

Notice that the appearance of the q-Meixner polynomials here as a dual family with respect to the big q-Laguerre polynomials is quite natural because the transformation $q \to q^{-1}$ interrelates these two sets of polynomials, that is,

$$M_n(x;b,c;q^{-1}) = (q^{-n}/b;q)_n\, P_n(qx/b;1/b,-c;q).$$

Let us introduce the Hilbert space $\mathfrak{l}^2_{a,b}$ of functions $F(q^{-m})$, defined on the set $m \in \{0, 1, 2, \cdots\}$, with a scalar product given by the formula

$$\langle F_1, F_2 \rangle_b = \sum_{m=0}^{\infty} \rho(m)\, F_1(q^{-m})\, \overline{F_2(q^{-m})}, \tag{6.27}$$

where the weight function

$$\rho(m) = \frac{(aq,bq;q)_m}{(q;q)_m(-abq^2)^m} q^{-m(m-1)/2}$$

is the same as in (6.22)–(6.24). Now we can formulate the following statement.

THEOREM 6.1. *The functions* (6.20) *and* (6.21) *constitute an orthogonal basis in the Hilbert space* $\mathfrak{l}^2_{a,b}$.

Proof. To show that the system of functions (6.20) and (6.21) constitutes a complete basis in the space $\mathfrak{l}^2_{a,b}$ we take in $\mathfrak{l}^2_{a,b}$ the set of functions F_k, $k = 0, 1, 2, \cdots$, such that $F_k(q^{-m}) = \delta_{km}$. It is clear that these functions constitute a basis in the space $\mathfrak{l}^2_{a,b}$. Let us show that each of these functions F_k belongs to the closure $\bar{V}$ of the linear span V of the functions (6.20) and (6.21). This will prove the theorem 6.1. We consider the functions

$$\hat{F}_k(q^{-m}) = \sum_{n=0}^{\infty} \hat{b}_{kn}\hat{b}_{mn} + \sum_{n=0}^{\infty} \hat{b}'_{kn}\,\hat{b}'_{mn}, \quad k = 0, 1, 2, \cdots,$$

where $\hat{b}_{jn}$ and $\hat{b}'_{kn}$ are the same as in (6.18). Then $\rho^{-1}(m)\hat{F}_k(q^{-m})$ is an infinite linear combination of the functions (6.20) and (6.21). Moreover, $\hat{F}_k(q^{-m}) \in \bar{V}$ and, due to (6.18), $\hat{F}_k$,

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$k=0,1,2,\cdots$, coincide, up to a constant, with the corresponding functions F_k, introduced above. The theorem 6.1 is proved. $\square$

The weight function $\rho(m)$ in (6.27) does not coincide with the orthogonality measure for q-Meixner polynomials. Multiplying this weight function by

$$[(bq;q)_m(-b)^{-m}q^{-m(m+1)/2}]^{-2},$$

we obtain the measure in (6.25). Let $\mathfrak{l}^2_{(1)}$ be the Hilbert space of functions $F(q^{-m})$ on the set $m\in\{0,1,2,\cdots\}$ with the scalar product

$$\langle F_1,F_2\rangle_{(1)}=\sum_{m=0}^{\infty}\frac{(aq;q)_m(-b/a)^m}{(bq,q;q)_m}\,q^{m(m-1)/2}\,F_1(q^{-m})\,\overline{F_2(q^{-m})},$$

where the weight function coincides with the measure in (6.25).

Taking into account the modification of the measure and the statement of Theorem 6.1, we conclude that the q-Meixner polynomials $M_n(q^{-m};\,a,-b/a;\,q)$ and the functions

$$(bq;q)_m\,(-b)^{-m}\,q^{-m(m+1)/2}\,g_n(q^{-m};a,b|q)$$

constitute an orthogonal basis in the space $\mathfrak{l}^2_{(1)}$.

PROPOSITION 6.2. *The q-Meixner polynomials $M_n(q^{-m};a,c;q)$, $n=0,1,2,\cdots$, with the parameters a and $c=-b/a$ do not constitute a complete basis in the Hilbert space $\mathfrak{l}^2_{(1)}$, that is, the q-Meixner polynomials are associated with the indeterminate moment problem and the measure in* (6.25) *is not an extremal measure for these polynomials.*

Proof. In order to prove this proposition we note that if the q-Meixner polynomials were associated with the determinate moment problem, then they would constitute a basis in the space of square integrable functions with respect to the measure from (6.25). However, this is not the case. By the definition of an extremal measure, if the measure in (6.25) were extremal, then again the set of the q-Meixner polynomials would be a basis in that space. Therefore, the measure is not extremal. Proposition is proved. $\square$

Let now $\mathfrak{l}^2_{(2)}$ be the Hilbert space of functions $F(q^{-m})$ on the set $m\in\{0,1,2,\cdots\}$, with the scalar product

$$\langle F_1,F_2\rangle_{(2)}=\sum_{m=0}^{\infty}\frac{(bq;q)_m(-a/b)^m}{(aq,q;q)_m}\,q^{m(m-1)/2}\,F_1(q^{-m})\,\overline{F_2(q^{-m})}.$$

The measure here coincides with the orthogonality measure in (6.26) for q-Meixner polynomials $M_n(q^{-m};b,-a/b;\,q)$, $b<0$. The following proposition is proved in the same way as Proposition 6.2.

PROPOSITION 6.3. *The q-Meixner polynomials $M_n(q^{-m};b,-a/b;q)$, $n=0,1,\cdots$, with $b<0$ do not constitute a complete basis in the Hilbert space $\mathfrak{l}^2_{(2)}$, that is, these q-Meixner polynomials are associated with the indeterminate moment problem and the measure in* (6.26) *is not an extremal measure for them.*

6.4. Generating function for big q-Laguerre polynomials. The aim of this section is to derive a generating function for the big q-Laguerre polynomials

$$G(x,t;a,b;\,q):=\sum_{n=0}^{\infty}\frac{(aq,bq;q)_n}{(q;q)_n}q^{-n(n-1)/2}P_n(x;a,b;q)t^n,$$

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which will be used in the next subsection. Observe that this formula is a bit more general than each of the three instances of generating functions for big q-Laguerre polynomials, given in section 3.11 of [27].

Employing the explicit expression

$$P_n(x;a,b;q) = (b^{-1}q^{-n};q)_n^{-1}\,{}_2\phi_1(q^{-n}, aqx^{-1};\; aq;\; q, x/q)$$

for big q-Laguerre polynomials, one obtains

$$G(x,t;a,b;\,q) = \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n}\,(-bqt)^n \sum_{k=0}^{n} \frac{(q^{-n}, aqx^{-1};q)_k}{(aq,q;q)_k}\,\left(\frac{x}{b}\right)^k$$

$$= \sum_{n=0}^{\infty} (aq;q)_n\,(-bqt)^n \sum_{k=0}^{n} \frac{(-x/b)^k\,(aqx^{-1};q)_k}{(aq,q;q)_k\,(q;q)_{n-k}}\,q^{-nk+k(k-1)/2}$$

$$= \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k(-x/b)^k}{(aq,q;q)_k}\,q^{k(k-1)/2} \sum_{m=0}^{\infty} \frac{(aq;q)_{m+k}}{(q;q)_m}\,(-bqt)^{m+k}\,q^{-(k+m)k}$$

$$= \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k}{(q;q)_k}\,(xt)^k\,q^{-k(k-1)/2} \sum_{m=0}^{\infty} \frac{(aq^{k+1};q)_m}{(q;q)_m}\,(-bq^{1-k}t)^m.$$

By the q-binomial theorem, the last sum equals to $(-abq^2;q)_\infty/(-bq^{1-k}t;q)_\infty$. Since

$$(-bq^{1-k}t;q)_\infty = q^{-k(k-1)/2}\,(-q/bqt;q)_k\,(-bqt;q)_\infty,$$

then

$$\frac{(-abq^2;q)_\infty}{(-bq^{1-k}t;q)_\infty} = \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty}\,\frac{q^{k(k-1)/2}}{(bt)^k(-1/bt;q)_k}.$$

Thus,

$$G(x,t;a,b;\,q) = \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty} \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k}{(-1/bt,q;q)_k}\left(\frac{x}{b}\right)^k$$

$$= \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty}\,{}_2\phi_1(aqx^{-1},\,0;\,-1/bt;\;q, x/b). \tag{6.28}$$

This gives a desired generating function for big q-Laguerre polynomials.

6.5. Biorthogonal systems of functions. Note that the operator A from formula (6.1) can be written in the form

$$A = \alpha\,q^{J_0/4}\,\left(\sqrt{1-bQ}\,J_+\,Q^{1/2} + Q^{1/2}\,J_-\,\sqrt{1-bQ}\right)\,q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 Q,$$

where

$$\alpha = (-abq)^{1/2}(1-q), \quad \beta_1 = b(1+q), \quad \beta_2 = bq + aq(b+1),$$

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and $J_{\pm}$, Q and q^{J_0} are the operators on $\mathcal{H}$ given as

$$J_+\, f_n = \frac{a^{-1/4} q^{-n/2}}{1-q}\, \sqrt{(1-q^{n+1})(1-aq^{n+1})}\, f_{n+1},$$

$$J_-\, f_n = \frac{a^{-1/4} q^{-(n-1)/2}}{1-q}\, \sqrt{(1-q^{n})(1-aq^{n})}\, f_{n-1},$$

$$q^{J_0}\, f_n = q^n\, (aq)^{1/2}\, f_n, \quad Q\, f_n = q^n\, f_n.$$

From the very beginning we could consider an operator

$$A_1 := \alpha\, q^{J_0/4}\, [(1-bQ)\, J_+ + Q\, J_-]\, q^{J_0/4} - \beta_1\, q^{2J_0} + \beta_2\, Q,$$

where α, β_1, and β_2 are the same as the above. This operator is well defined, but it is not self-adjoint. Repeating the reasoning of section 3, we find that eigenfunctions of A_1 are of the form

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} (-ab)^{-n/2}\, q^{-n} \left(\frac{(aq;q)_n}{(q;q)_n} \right)^{1/2} P_n(\lambda; a, b; q)\, f_n(x)$$

$$= \sum_{n=0}^{\infty} a^{-3n/4}\, (-b)^{-n/2}\, q^{-n}\, \frac{(aq;q)_n}{(q;q)_n}\, P_n(\lambda; a, b; q)\, x^n. \tag{6.29}$$

The last sum can be summed with the aid of formula (3.11.12) in [27]. We thus have

$$\psi_\lambda(x) = ((-a/b^2)^{1/4} x; q)_\infty \cdot {}_2\phi_1(bq\lambda^{-1},\, 0;\, bq;\; q, a^{-3n/4}(-b)^{-1/2} q^{-1} x\lambda)\,.$$

Now we consider another operator

$$A_2 := \alpha\, q^{J_0/4}\, [J_+\, Q + J_-\, (1-bQ)]\, q^{J_0/4} - \beta_1\, q^{2J_0} + \beta_2\, Q.$$

This operator is adjoint to the operator $A_1 : A_2^* = A_1$. Repeating the reasoning of subsection 6.1, we find that eigenfunctions of A_2 have the form

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty} (-ab)^{-n/2}\, q^{-n(n+1)/2} \left(\frac{(aq;q)_n (bq;q)_n^2}{(q;q)_n} \right)^{1/2} P_n(\lambda; a, b; q)\, f_n(x)$$

$$= \sum_{n=0}^{\infty} a^{-3n/4}\, (-b)^{-n/2}\, q^{-n(n+1)/2}\, \frac{(aq;q)_n (bq;q)_n}{(q;q)_n}\, P_n(\lambda; a, b; q)\, x^n. \tag{6.30}$$

According to the formula (6.28), this function can be written as

$$\varphi_\lambda(x) = \frac{(-abq^2;q)_\infty}{(a^{-3/4}\,(-b)^{1/2} x;q)_\infty}\, {}_2\phi_1(aq/\lambda, 0;\, a^{3/4}(-b)^{-1/2} q/x;\; q, \lambda/b).$$

Let us denote by $\Psi_m(x)$, $m = 0, \pm 1, \pm 2, \cdots$, the functions

$$\Psi_m(x) = c_m \psi_{aq^{m+1}}(x), \;\; m = 0, 1, 2, \cdots, \quad \Psi_{-m}(x) = c'_{m-1} \psi_{bq^m}(x), \;\; m = 1, 2, \cdots,$$

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and by $\Phi_m(x)$, $m = 0, \pm1, \pm2, \cdots$, the functions

$$\Phi_m(x) = c_m \varphi_{aq^{m+1}}(x), \;\; m = 0, 1, 2, \cdots, \quad \Phi_{-m}(x) = c'_{m-1} \varphi_{bq^m}(x), \;\; m = 1, 2, \cdots,$$

where c_m and c'_m are given by formulas (6.9) and (6.10).

Writing down the decompositions (6.29) and (6.30) for the functions $\Psi_m(x)$ and $\Phi_m(x)$ (in terms of the orthonormal basis f_n, $n = 0, 1, 2, \cdots$, of the Hilbert space $\mathcal{H}$) and taking into account the orthogonality relations (6.22)–(6.24) we find that

$$\langle\, \Psi_m(x), \Phi_n(x)\rangle = \delta_{mn}, \quad m, n = 0, \pm1, \pm2, \cdots .$$

This means that we can formulate the following statement.

THEOREM 6.4. *The set of functions $\Psi_m(x)$, $m = 0, \pm1, \pm2, \cdots$, and the set of functions $\Phi_m(x)$, $m = 0, \pm1, \pm2, \cdots$, form biorthogonal sets of functions with respect to the scalar product in the Hilbert space $\mathcal{H}$.*

7. Alternative q-Charlier polynomials and their duals.

7.1. Pair of operators (B_1, J). Let $\mathcal{H}$ be the same separable complex Hilbert space as before. We have introduced into this space the orthonormal basis f_n, $n = 0, 1, 2, \cdots$, expressed in terms of monomials in x. We define on $\mathcal{H}$ two operators. The first one, denoted as Q, acts on the basis elements as

$$Q\, f_n = q^n\, f_n.$$

The second one, denoted as B_1, is given by the formula

$$B_1\, f_n = a_n\, f_{n+1} + a_{n-1}\, f_{n-1} + b_n\, f_n \tag{7.1}$$

with

$$a_n = -(aq^{3n+1})^{1/2} \frac{\sqrt{(1-q^{n+1})(1+aq^n)}}{(1+aq^{2n+1})\sqrt{(1+aq^{2n})(1+aq^{2n+2})}}\,,$$

$$b_n = q^n \left(\frac{1+aq^n}{(1+aq^{2n})(1+aq^{2n+1})} + aq^{n-1} \frac{1-q^n}{(1+aq^{2n-1})(1+aq^{2n})} \right),$$

where a is a fixed positive number. Clearly, B_1 is a symmetric operator.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator B_1 is bounded. We assume that it is defined on the whole Hilbert space $\mathcal{H}$. For this reason, B_1 is a self-adjoint operator. Let us show that B_1 is a Hilbert–Schmidt operator. For the coefficients a_n and b_n from (7.1), we have $a_{n+1}/a_n \to q^{3/2}$ and $b_{n+1}/b_n \to q$ when $n \to \infty$. Since $0 < q < 1$, for the sum of all matrix elements of the operator B_1 in the basis f_n, $n = 0, 1, 2, \cdots$, we have $\sum_n (2a_n + b_n) < \infty$. This means that B_1 is a Hilbert–Schmidt operator. Thus, a spectrum of B_1 is discrete and has a single accumulation point at 0. Moreover, a spectrum of B_1 is simple, since B_1 is representable by a Jacobi matrix with $a_n \neq 0$ (see [15], Chapter VII).

To find eigenfunctions ξ_λ of the operator B_1, $B_1\, \xi_\lambda = \lambda \xi_\lambda$, we set

$$\xi_\lambda = \sum_n \beta_n(\lambda)\, f_n,$$

where $\beta_n(\lambda)$ are appropriate numerical coefficients. Acting by the operator B_1 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda)\; (a_n\, f_{n+1} + a_{n-1}\, f_{n-1} + b_n\, f_n) = \lambda \sum_{n=0}^{\infty} \beta_n(\lambda)\, f_n,$$

where a_n and b_n are the same as in (7.1). Collecting in this identity all factors, which multiply f_n with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)\, a_n + \beta_{n-1}(\lambda)\, a_{n-1} + \beta_n(\lambda)\, b_n = \lambda\, \beta_n(\lambda).$$

The substitution

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n\,(1+aq^{2n})}{(q;q)_n\,(1+a)(a/q)^n} \right)^{1/2} q^{-n(n+3)/4}\, \beta'_n(\lambda)$$

reduces this relation to the following one

$$-A_n\, \beta'_{n+1}(\lambda) - C_n\, \beta'_{n-1}(\lambda) + (A_n + C_n)\, \beta'_n(\lambda) = \lambda\, \beta'_n(\lambda),$$

$$A_n = q^n\, \frac{1+aq^n}{(1+aq^{2n})(1+aq^{2n+1})}, \qquad C_n = a\, q^{2n-1}\, \frac{1-q^n}{(1+aq^{2n-1})(1+aq^{2n})}.$$

This is the recurrence relation for the alternative q-Charlier polynomials

$$K_n(\lambda; a; q) := {}_2\phi_1(q^{-n}, -aq^n;\; 0;\; q, q\lambda)$$

(see, formulas (3.22.1) and (3.22.2) in [27]). Therefore, $\beta'_n(\lambda) = K_n(\lambda; a; q)$ and

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n\,(1+aq^{2n})}{(q;q)_n\,(1+a)a^n} \right)^{1/2} q^{-n(n+1)/4}\, K_n(\lambda; a; q). \tag{7.2}$$

For the eigenvectors ξ_λ we thus have the expression

$$\xi_\lambda = \sum_{n=0}^{\infty} \left(\frac{(-a;q)_n\,(1+aq^{2n})}{(q;q)_n\,(1+a)\,a^n} \right)^{1/2} q^{-n(n+1)/4}\, K_n(\lambda; a; q)\, f_n. \tag{7.3}$$

Since the spectrum of the operator B_1 is discrete, only for a discrete set of values of λ these vectors belong to the Hilbert space $\mathcal{H}$.

Now we look for a spectrum of the operator B_1 and for a set of polynomials, dual to alternative q-Charlier polynomials. To this end we use the action of the operator

$$J := Q^{-1} - a\, Q$$

upon the eigenvectors ξ_λ, which belong to the Hilbert space $\mathcal{H}$. In order to find how this operator acts upon these vectors, one can use the q-difference equation

$$(q^{-n} - aq^n)\, K_n(\lambda) = -a\, K_n(q\lambda) + \lambda^{-1}\, K_n(\lambda) - \lambda^{-1}(1-\lambda)\, K_n(q^{-1}\lambda) \tag{7.4}$$

for the alternative q-Charlier polynomials $K_n(\lambda) \equiv K_n(\lambda; a; q)$ (see formula (3.22.5) in [27]). Multiply both sides of (7.4) by $k_n\, f_n$ and sum up over n, where k_n are the coefficients of the $K_n(\lambda; a; q)$ in the expression (7.2) for $\beta_n(\lambda)$. Taking into account formula (7.3) and the fact that $J f_n = (q^{-n} - aq^n) f_n$, one obtains the relation

$$J\, \xi_\lambda = -a\, \xi_{q\lambda} + \lambda^{-1}\, \xi_\lambda - \lambda^{-1}(1-\lambda)\, \xi_{q^{-1}\lambda}, \tag{7.5}$$

which will be used in the next subsection.

7.2. Spectrum of B_1 and orthogonality of alternative q-Charlier polynomials. The aim of this section is to find, by using the operators B_1 and J, a basis in the Hilbert space $\mathcal{H}$, which consists of eigenfunctions of the operator B_1 in a normalized form, and to derive explicitly the unitary matrix U, connecting this basis with the basis $f_n, n = 0, 1, 2, \cdots$, in $\mathcal{H}$. This matrix leads directly to the orthogonality relation for alternative q-Charlier polynomials. For this purpose we first find the spectrum of B_1.

We proceed as in the previous cases. First we analyze a form of the spectrum of the operator B_1 from the point of view of the spectral theory of Hilbert–Schmidt operators. If λ is a spectral point of the operator B_1, then (as it is easy to see from (7.5)) a successive action by the operator J upon the function (eigenfunction of B_1) ξ_λ leads to the eigenfunctions $\xi_{q^m\lambda}$, $m = 0, \pm 1, \pm 2, \cdots$. However, since B_1 is a Hilbert–Schmidt operator, not all of these points may belong to the spectrum of B_1, since $q^{-m}\lambda \to \infty$ when $m \to +\infty$ once $\lambda \neq 0$. This means that the coefficient $1 - \lambda'$ of $\xi_{q^{-1}\lambda'}$ in (7.5) must vanish for some eigenvalue λ'. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for the coefficient of $\xi_{q^{-1}\lambda'}$ in (7.5) to vanish, that is, the point $\lambda = 1$ is a spectral point for the operator B_1. Let us show that the corresponding eigenfunction $\xi_1 \equiv \xi_{q^0}$ belongs to the Hilbert space $\mathcal{H}$.

By formula (II.6) of Appendix II in [21], one has

$$K_n(1; a; q) = {}_2\phi_1(q^{-n}, -aq^n;\ 0;\ q, q) = (-a)^n q^{n^2}.$$

Therefore,

$$\langle \xi_1, \xi_1 \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n (1 + aq^{2n})}{(1 + a)(q; q)_n\, a^n\, q^{n(n+1)/2}}\ K_n^2(1; a; q)$$

$$= \sum_{n=0}^{\infty} \frac{(-a; q)_n\, (1 + aq^{2n})}{(1 + q)\, (q; q)_n}\, q^{n(3n-1)/2}\, a^n. \tag{7.6}$$

In order to calculate this sum, we take the limit $d, e \to \infty$ in the equality

$$\sum_{n=0}^{\infty} \frac{(1 + aq^{2n})(-a; q)_n (d; q)_n (e; q)_n}{(1 + a)(-aq/d; q)_n (-aq/e; q)_n (q; q)_n} \left(\frac{aq}{de}\right)^n q^{n(n-1)/2} = \frac{(-aq; q)_\infty (-aq/de; q)_\infty}{(-aq/d; q)_\infty (-aq/e; q)_\infty}$$

(see formula in Exercise 2.12, Chapter 2 of [21]). Since

$$\lim_{d,e\to\infty} (d; q)_n\, (e; q)_n\, (aq/de)^n = q^{n^2}\, a^n,$$

we obtain from here that the sum in (7.6) is equal to $(-aq; q)_\infty$, that is, $\langle \xi_1, \xi_1 \rangle < \infty$ and ξ_1 belongs to the Hilbert space $\mathcal{H}$. Thus, the point $\lambda = 1$ does belong to the spectrum of the operator B_1.

Let us find other spectral points of the operator B_1 (recall that the spectrum of B_1 is discrete). Setting $\lambda = 1$ in (7.5), we see that the operator J transforms ξ_{q^0} into a linear combination of the vectors ξ_q and ξ_{q^0}. Moreover, ξ_q belongs to the Hilbert space $\mathcal{H}$, since the series

$$\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n\, (1 + aq^{2n})}{(1 + a)(q; q)_n\, a^n}\, q^{-n(n+1)/2}\, K_n^2(q; a; q)$$

is majorized by the corresponding series (7.6) for ξ_{q^0}. Therefore, ξ_q belongs to the Hilbert space $\mathcal{H}$ and the point q is an eigenvalue of the operator B_1. Similarly, setting $\lambda = q$ in

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(7.5), we find that ξ_{q^2} is an eigenvector of B_1 and the point q^2 belongs to the spectrum of B_1. Repeating this procedure, we find that all ξ_{q^n}, $n = 0, 1, 2, \cdots$, are eigenvectors of B_1 and the set q^n, $n = 0, 1, 2, \cdots$, belongs to the spectrum of B_1. So far, we do not know yet whether other spectral points exist or not.

The functions ξ_{q^n}, $n = 0, 1, 2, \cdots$, are linearly independent elements of the Hilbert space $\mathcal{H}$ (since they correspond to distinct eigenvalues of the self-adjoint operator B_1). Suppose that values q^n, $n = 0, 1, 2, \cdots$, constitute a whole spectrum of B_1. Then the set of vectors ξ_{q^n}, $n = 0, 1, 2, \cdots$, is a basis in the Hilbert space $\mathcal{H}$. Introducing the notation $\Xi_k := \xi_{q^k}$, $k = 0, 1, 2, \cdots$, we find from (7.5) that

$$J\,\Xi_k = -a\,\Xi_{k+1} + q^{-k}\,\Xi_k - q^{-k}\,(1-q^k)\,\Xi_{k-1}.$$

As we see, the matrix of the operator J in the basis Ξ_k, $k = 0, 1, 2, \cdots$, is not symmetric, although in the initial basis $|n\rangle$, $n = 0, 1, 2, \cdots$, it was symmetric. The reason is that the matrix (a_{mn}) with entries $a_{mn} := \beta_m(q^n)$, $m, n = 0, 1, 2, \cdots$, where $\beta_m(q^n)$ are the coefficients (7.2) in the expansion $\xi_{q^n} = \sum_m \beta_m(q^n)\, f_n$, is not unitary. This fact is equivalent to the statement that the basis $\Xi_n = \xi_{q^n}$, $n = 0, 1, 2, \cdots$, is not normalized. To normalize it, one has to multiply Ξ_n by corresponding numbers c_n. Let $\hat{\Xi}_n = c_n \Xi_n$, $n = 0, 1, 2, \cdots$, be a normalized basis. Then the matrix of the operator J is symmetric in this basis. Since J has in the basis $\{\hat{\Xi}_n\}$ the form

$$J\,\hat{\Xi}_n = -c_{n+1}^{-1}\,c_n a\,\hat{\Xi}_{n+1} + q^{-n}\,\hat{\Xi}_n - c_{n-1}^{-1} c_n q^{-n}(1-q^n)\,\hat{\Xi}_{n-1},$$

then its symmetricity means that $c_{n+1}^{-1} c_n a = c_n^{-1} c_{n+1} q^{-n-1}(1-q^{n+1})$, that is, $c_n/c_{n-1} = \sqrt{aq^n/(1-q^n)}$. Therefore,

$$c_n = c(a^n q^{n(n+1)/2}/(q;q)_n)^{1/2},$$

where c is a constant.

The expansions

$$\hat{\xi}_{q^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\,\beta_m(q^n)|m\rangle \equiv \sum_m \hat{a}_{mn}|m\rangle$$

connect two orthonormal bases in the Hilbert space $\mathcal{H}$. This means that the matrix $(\hat{a}_{mn})$, $m, n = 0, 1, 2, \cdots$, with entries

$$\hat{a}_{mn} = c_n\,\beta_m(q^n) = c\left(\frac{a^n q^{n(n+1)/2}}{(q;q)_n}\,\frac{(-a;q)_m\,(1+aq^{2m})}{(1+a)(q;q)_m\,a^m q^{m(m+1)/2}}\right)^{1/2} K_m(q^n;a;q)$$

is unitary, provided that the constant c is appropriately chosen. In order to calculate this constant, we use the relation $\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1$ for $n = 0$. Then this sum is a multiple of the sum in (7.6) and, consequently,

$$c = (-aq;q)_\infty^{-1/2}.$$

The matrix $(\hat{a}_{mn})$ is real and orthogonal, that is,

$$\sum_n \hat{a}_{mn}\,\hat{a}_{m'n} = \delta_{mm'}, \quad \sum_m \hat{a}_{mn}\,\hat{a}_{mn'} = \delta_{nn'}. \tag{7.7}$$

Substituting into the first sum over n in (7.7) the expressions for $\hat{a}_{mn}$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n}\,K_m(q^n;a;q)\,K_{m'}(q^n;a;q)$$

$$= \frac{(-aq^m;q)_\infty\, a^m\, (q;q)_m}{(1+aq^{2m})}\, q^{m(m+1)/2}\delta_{mm'}\,, \tag{7.8}$$

which must yield the orthogonality relation for alternative q-Charlier polynomials. An only gap, which remains to be clarified, is the following. We have assumed that the points q^n, $n=0,1,2,\cdots$, exhaust the whole spectrum of B_1. Let us show that this is the case.

Recall that the self-adjoint operator I_1 is represented by a Jacobi matrix in the basis f_n, $n=0,1,2,\cdots$. According to the theory of operators of such type, eigenvectors ξ_λ of B_1 are expanded into series in the basis f_n, $n=0,1,2,\cdots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some positive measure $d\mu(\lambda)$ (moreover, for self-adjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which these polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple.

We have found that the spectrum of B_1 contains the points q^n, $n=0,1,2,\cdots$. If the operator B_1 had other spectral points x, then on the left-hand side of (7.8) there would be other summands $\mu_{x_k}\, K_m(x_k;a;q)\, K_{m'}(x_k;a;q)$, corresponding to these additional points. Let us show that these additional summands do not appear. We set $m=m'=0$ in the relation (7.8) with the additional summands. Since $K_0(x;a;q)=1$, we have the equality

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} + \sum_k \mu_{x_k} = (-aq;q)_\infty.$$

According to the formula for the q-exponential function $E_q(a)$ (see formula (II.2) of Appendix II in [21]), we have $\sum_{n=0}^{\infty} a^n\, q^{n(n+1)/2}/(q;q)_n = (-aq;q)_\infty$. Hence, $\sum_k \mu_{x_k}=0$. This means that additional summands do not appear in (7.8) and it does represent the orthogonality relation for alternative q-Charlier polynomials.

Due to the orthogonality relation for the alternative q-Charlier polynomials, we arrive at the following statement:

PROPOSITION 7.1. *The spectrum of the operator B_1 coincides with the set of points q^n, $n=0,1,2,\cdots$. The spectrum is simple and has one accumulation point at 0.*

7.3. Dual alternative q-Charlier polynomials. Now we consider the second identity in (7.7), which gives the orthogonality relation for the matrix elements $\hat{a}_{mn}$, considered as functions of m. Up to multiplicative factors these functions coincide with

$$F_n(x;a|q) = {}_2\phi_1(x,-a/x;\;0;\;q,q^{n+1}),$$

considered on the set $x\in\{q^{-m}\,|\,m=0,1,2,\cdots\}$. Consequently,

$$\hat{a}_{mn} = \left(\frac{a^n q^{n(n+1)/2}}{(q;q)_n}\frac{(1+aq^{2m})}{(-aq^m;q)_\infty (q;q)_m\, a^m q^{m(m+1)/2}}\right)^{1/2} F_n(q^{-m};a|q)$$

and the second identity in (7.7) gives the orthogonality relation for $F_n(q^{-m};a|q)$:

$$\sum_{m=0}^{\infty} \frac{(1+aq^{2m})}{a^m(-aq^m;q)_\infty (q;q)_m}\, q^{-m(m+1)/2}\, F_n(q^{-m};a|q)\, F_{n'}(q^{-m};a|q)$$

$$= \frac{(q;q)_n}{a^n}\, q^{-n(n+1)/2}\, \delta_{nn'}\,. \tag{7.9}$$

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The functions $F_n(x;a,b|q)$ can be represented in another form. Indeed, taking in the relation (III.8) from Appendix III in [21] the limit $c\to\infty$, one derives the relation

$$ {}_2\phi_1(q^{-m},-aq^m;\ 0;\ q,q^{n+1}) = (-a)^m q^{m^2}\,{}_3\phi_0(q^{-m},-aq^m,q^{-n}\ -;\ q,-q^n/a). $$

Therefore, we have

$$ F_n(q^{-m};a|q) = (-a)^m\,q^{m^2}\,{}_3\phi_0(q^{-m},-aq^m,q^{-n};\ -;\ q,-q^n/a). \tag{7.10} $$

The basic hypergeometric function ${}_3\phi_0$ in (7.10) is a polynomial of degree n in the variable $\mu(m):=q^{-m}-a\,q^m$, which represents a q-quadratic lattice; we denote it by

$$ d_n(\mu(m);a;q) := {}_3\phi_0(q^{-m},-a\,q^m,q^{-n};\ -;\ q,-q^n/a)\,. \tag{7.11} $$

Then formula (7.9) yields the orthogonality relation

$$ \sum_{m=0}^{\infty}\frac{(1+aq^{2m})a^m}{(-aq^m;q)_\infty(q;q)_m}\,q^{m(3m-1)/2}\,d_n(\mu(m);a;q)\,d_{n'}(\mu(m);a;q) $$

$$ = \frac{(q;q)_n}{a^n}\,q^{-n(n+1)/2}\,\delta_{nn'} \tag{7.12} $$

for the polynomials (7.11) when $a>0$. We call the polynomials $d_n(\mu(m);a;q)$ *dual alternative q-Charlier polynomials*. Thus, the following theorem holds.

THEOREM 7.2. *The polynomials $d_n(\mu(m);a;q)$, given by formula (7.11), are orthogonal on the set of points $\mu(m):=q^{-m}-a\,q^m$, $m=0,1,2,\cdots$, and the orthogonality relation is given by formula (7.12).*

Let $\mathfrak{l}^2$ be the Hilbert space of functions on the set $m=0,1,2,\cdots$ with the scalar product

$$ \langle F_1,F_2\rangle = \sum_{m=0}^{\infty}\frac{(1+aq^{2m})a^m}{(-aq^m;q)_\infty(q;q)_m}\,q^{m(3m-1)/2}\,F_1(m)\,\overline{F_2(m)}, \tag{7.13} $$

where the weight function is taken from (7.12). The polynomials (7.11) are in one-to-one correspondence with the columns of the orthogonal matrix $(\hat a_{mn})$ and the orthogonality relation (7.12) is equivalent to the orthogonality of these columns. Due to (7.7) the columns of the matrix $(\hat a_{mn})$ form an orthonormal basis in the Hilbert space of sequences $\mathbf{a}=\{a_n\,|\,n=0,1,2,\cdots\}$ with the scalar product $\langle\mathbf{a},\mathbf{a}'\rangle=\sum_n a_n\overline{a'_n}$. This scalar product is equivalent to the scalar product (7.13) for the polynomials $d_n(\mu(m);a;q)$. For this reason, the set of polynomials $d_n(\mu(m);a;q)$, $n=0,1,2,\cdots$, form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$. This means that *either the dual alternative q-Charlier polynomials $d_n(\mu(m);a;q)$ correspond to determinate moment problem or the point measure in* (7.12) *is extremal if these polynomials correspond to indeterminate moment problem.* This question will not be further pursued here.

A recurrence relation for the polynomials $d_n(\mu(m);a;q)$ is derived from (7.4). It has the form

$$ (q^{-m}-a\,q^m)\,d_n(\mu(m)) \tag{7.14} $$

$$ = -a\,d_{n+1}(\mu(m)) + q^{-n}\,d_n(\mu(m)) - q^{-n}(1-q^n)\,d_{n-1}(\mu(m)), $$

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where $d_n(\mu(m)) \equiv d_n(\mu(m); a; q)$. A q-difference equation for $d_n(\mu(m); a; q)$ can be obtained from the three-term recurrence relation for alternative q-Charlier polynomials.

Note that for the polynomials $d_n(\mu(m); a; q^{-1})$ with $q < 1$ we have the expression

$$d_n(\mu(m); a; q^{-1}) = {}_3\phi_2(q^{-m}, -a\, q^m, q^{-n};\ 0, 0;\ q, q). \tag{7.15}$$

However, the recurrence relation for these polynomials (which can be obtained from the relation (7.14)), does not satisfy the positivity condition $A_n\, C_{n+1} > 0$, that is, they are not orthogonal polynomials for $a > 0$ (as it is the case for alternative q-Charlier polynomials). This positivity condition holds only if we require that $a < 0$. In this case, the polynomials (7.15) are the continuous big q-Hermite polynomials $H_n(x; a|q)$ (for an explicit form of these polynomials see, for example, [27], formula (3.18.1)), which are orthogonal on a certain continuous set.

8. Duality of Al-Salam–Carlitz I and q-Charlier polynomials.

8.1. Pair of operators (B_2, Q^{-1}).

Let a be a real number such that $a < 0$. Let $\mathcal{L}$ be the separable complex Hilbert space with the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \cdots$. We define on $\mathcal{L}$ the operator B_2, which is given by the formula

$$B_2|n\rangle = a_n\,|n+1\rangle + a_{n-1}\,|n-1\rangle - b_n\,|n\rangle\,, \tag{8.1}$$

with

$$a_n = (-a)^{1/2} q^{n/2} \sqrt{1-q^{n+1}}, \quad b_n = (a+1)\, q^n.$$

Clearly, B_2 is a bounded symmetric operator. Therefore, we assume that it is defined on the whole Hilbert space $\mathcal{L}$. For this reason, B_2 is a self-adjoint operator. As in the previous cases, it is easy to show that B_2 is a Hilbert–Schmidt operator. Thus, the spectrum of B_2 is discrete and has a single accumulation point at 0. Moreover, the spectrum of B_2 is simple, since B_2 is representable by a Jacobi matrix with $a_n \neq 0$.

To find corresponding eigenfunctions ξ_λ of the operator B_2, $B_2\xi_\lambda = \lambda\xi_\lambda$, we set $\xi_\lambda = \sum_n \beta_n(\lambda)\,|n\rangle$, where $\beta_n(\lambda)$ are appropriate numerical coefficients. Acting by the operator B_2 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda)\ (a_n\,|n+1\rangle + a_{n-1}\,|n-1\rangle - b_n\,|n\rangle) = \lambda \sum_{n=0}^{\infty} \beta_n(\lambda)\,|n\rangle,$$

where a_n and b_n are the same as in (8.1). Collecting in this identity all factors, which multiply $|n\rangle$ with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)\, a_n + \beta_{n-1}(\lambda)\, a_{n-1} - \beta_n(\lambda)\, b_n = \lambda\, \beta_n(\lambda).$$

Making the substitution

$$\beta_n(\lambda) = \left(\frac{q^{-n(n-1)/2}}{(q;q)_n\,(-a)^n}\right)^{1/2} \beta'_n(\lambda),$$

we reduce this relation to the following one

$$\beta'_{n+1}(\lambda) + (-a)\, q^{n-1}(1-q^n)\, \beta'_{n-1}(\lambda) - (a+1)\, q^n\, \beta'_n(\lambda) = \lambda\, \beta'_n(\lambda).$$

This is the recurrence relation for the Al-Salam–Carlitz I polynomials

$$U_n^{(a)}(\lambda; q) := (-a)^n\, q^{n(n-1)/2}\, {}_2\phi_1(q^{-n}, \lambda^{-1};\ 0;\ q, q\lambda/a)$$

(see, formula (3.24.1) in [27]). Therefore, $\beta'_n(\lambda) = U_n^{(a)}(\lambda; q)$ and

$$\beta_n(\lambda) = \left(\frac{q^{-n(n-1)/2}}{(q;q)_n \, (-a)^n} \right)^{1/2} U_n^{(a)}(\lambda; q). \tag{8.2}$$

For the eigenvectors ξ_λ we thus have the expression

$$\xi_\lambda = \sum_{n=0}^{\infty} \left(\frac{q^{-n(n-1)/2}}{(q;q)_n \, (-a)^n} \right)^{1/2} U_n^{(a)}(\lambda; q) \, |n\rangle. \tag{8.3}$$

Since the spectrum of the operator B_2 is discrete, only for a discrete set of values of λ these vectors belong to the Hilbert space $\mathcal{L}$. This discrete set of eigenvectors determines the spectrum of B_2.

Let us find the spectrum of the operator B_2 and a set of polynomials, dual to Al-Salam–Carlitz I polynomials. For this purpose we use the operator Q^{-1}, which is diagonal in the basis $\{|n\rangle\}$, and is given as

$$Q^{-1} \, |n\rangle = q^{-n} \, |n\rangle \, .$$

We have to find how the operator Q^{-1} acts upon the eigenvectors ξ_λ, which belong to the Hilbert space $\mathcal{L}$. To this end, one can use the q-difference equation for Al-Salam–Carlitz I polynomials, which can be written as

$$\begin{aligned} q^{-n} \, U_n^{(a)}(\lambda; q) &= a \, q^{-1} \lambda^{-2} \, U_n^{(a)}(q\lambda; q) - \, d_\lambda \, U_n^{(a)}(\lambda; q) \\ &\quad + \, a \, \lambda^{-2} \, (1-\lambda)(1-\lambda/a) \, U_n^{(a)}(q^{-1}\lambda; q), \end{aligned} \tag{8.4}$$

where $d_\lambda = a \, (1+q) \, (1-\lambda)/q \, \lambda^2$.

Multiply both sides of (8.4) by $k_n \, |n\rangle$ and sum up over n, where k_n are the coefficients of $U_n^{(a)}(\lambda; q)$ in the expression (8.2) for $\beta_n(\lambda)$. Taking into account formula (8.4) and the fact that $Q^{-1} \, |n\rangle = q^{-n} \, |n\rangle$, one obtains the relation

$$Q^{-1} \, \xi_\lambda = a \, q^{-1} \lambda^{-2} \, \xi_{q\lambda} - d_\lambda \, \xi_\lambda + \, a \, \lambda^{-2} \, (1-\lambda/a)(1-\lambda) \, \xi_{q^{-1}\lambda}, \tag{8.5}$$

which is used in the next subsection.

8.2. Spectrum of B_2 and orthogonality of Al-Salam–Carlitz polynomials. Let us analyze a form of the spectrum of B_2. If λ is a spectral point of the operator B_2, then (as it is easy to see from (8.5)) a successive action by the operator Q^{-1} upon the vector (eigenvector of B_2) ξ_λ leads to the eigenvectors $\xi_{q^m\lambda}$, $m = 0, \pm 1, \pm 2, \cdots$. However, since B_2 is a Hilbert–Schmidt operator, not all of these points may belong to the spectrum of B_2, since $q^{-m}\lambda \to \infty$ when $m \to +\infty$ if $\lambda \neq 0$. This means that the coefficient of $\xi_{q^{-1}\lambda'}$ in (8.5) must vanish for some eigenvalue λ'. There are two such values of λ: $\lambda = 1$ and $\lambda = a$. Let us show that both of these points are spectral points of B_2. Observe that $U_n^{(a)}(1; q) = (-a)^n q^{n(n-1)/2}$ and $U_n^{(a)}(a; q) = (-1)^n q^{n(n-1)/2}$. Hence, for the scalar product $\langle \xi_1, \xi_1 \rangle$ we have the expression

$$\sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{(q;q)_n (-a)^n} (-a)^{2n} q^{n(n-1)} = \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{(q;q)_n} \, (-a)^n = (a;q)_\infty. \tag{8.6}$$

Similarly, for $\langle \xi_a, \xi_a \rangle$ one has the expression

$$\langle \xi_a, \xi_a \rangle = \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{(q;q)_n} \, (-a)^{-n} = (1/a; q)_\infty. \tag{8.7}$$

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Thus, the values $\lambda = 1$ and $\lambda = a$ are spectral points of the operator B_2.

Let us find other spectral points of B_2. Setting $\lambda = 1$ in (8.5), we see that the operator Q^{-1} transforms ξ_{q^0} into a linear combination of the vectors ξ_q and ξ_1. We have to show that ξ_q belongs to the Hilbert space $\mathcal{L}$, that is, that

$$\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{q^{n(n-1/2}}{(q;q)_n\,(-a)^n}\,(-a)^{2n}\,q^{n(n-1)/2}\,\left[U_n^{(a)}(q;q)\right]^2 < \infty\,.$$

It is made in the same way as in the case of the scalar product $\langle \psi_{aq^2}, \psi_{aq^2}\rangle$ in subsection 4.2. Therefore, ξ_q belongs to the Hilbert space $\mathcal{L}$ and the point q is an eigenvalue of the operator B_2. Similarly, setting $\lambda = q$ in (8.5), we find that ξ_{q^2} is an eigenvector of B_2 and the point q^2 belongs to the spectrum of B_2. Repeating this procedure, we find that all ξ_{q^n}, $n = 0,1,2,\cdots$, are eigenvectors of B_2 and the set q^n, $n = 0,1,2,\cdots$, belongs to the spectrum of B_2. Likewise, one concludes that the elements ξ_{aq^n}, $n = 0,1,2,\cdots$, are eigenvectors of B_2 and the set aq^n, $n = 0,1,2,\cdots$, belongs to the spectrum of B_2. So far, we do not know yet whether there are other spectral points or not.

The vectors ξ_{q^n} and ξ_{aq^n}, $n = 0,1,2,\cdots$, are linearly independent elements of the Hilbert space $\mathcal{L}$. Suppose that values q^n and aq^n, $n = 0,1,2,\cdots$, constitute a whole spectrum of B_2. Then the set of vectors ξ_{q^n} and ξ_{aq^n}, $n = 0,1,2,\cdots$, is a basis in the Hilbert space $\mathcal{L}$. Introducing the notations $\Xi_k := \xi_{q^k}$ and $\Xi'_k := \xi_{aq^k}$, $k = 0,1,2,\cdots$, we find from (8.5) that

$$Q^{-1}\,\Xi_n = a\,q^{-2n-1}\,\Xi_{n+1} - d_n\,\Xi_n + a\,q^{-2n}(1-q^n)(1-q^n/a)\,\Xi_{n-1},$$

$$Q^{-1}\,\Xi'_n = a^{-1}\,q^{-2n-1}\,\Xi'_{n+1} - d'_n\,\Xi'_n + a^{-1}\,q^{-2n}(1-q^n)(1-a\,q^n)\,\Xi'_{n-1},$$

where

$$d_n = a\,(1+q)(1-q^n)\,q^{-(2n+1)}, \qquad d'_n = (1+q)(1-a\,q^n)\,a^{-1}\,q^{-(2n+1)}\,.$$

As we see, the matrix of the operator Q^{-1} in the basis Ξ_n, Ξ'_n, $n = 0,1,2,\cdots$, is not symmetric, although in the initial basis $|m\rangle$, $m = 0,1,2,\cdots$, it was symmetric. The reason is that the matrix $M := ((a_{mn})_{m,n=0}^{\infty}\ \ (a'_{mn})_{m,n=0}^{\infty}) \equiv ((a_{mn})\ \ (a'_{mn}))$ with entries

$$a_{mn} := \beta_m(q^n), \quad a'_{mn} := \beta_m(aq^n), \quad m,n = 0,1,2,\cdots\,,$$

where $\beta_m(dq^n)$, $d = 1, a$, are the coefficients (8.2) in the expansion

$$\xi_{dq^n} = \sum_m \beta_m(dq^n)\,|n\rangle\,,$$

is not unitary. This fact is equivalent to the statement that the basis $\Xi_n = \xi_{q^n}$, $\Xi'_n = \xi_{aq^n}$, $n = 0,1,2,\cdots$, is not normalized. To normalize it, one has to multiply Ξ_n by corresponding numbers c_n and Ξ'_n by corresponding numbers c'_n. Let $\hat{\Xi}_n = c_n\Xi_n$ and $\hat{\Xi}'_n = c'_n\Xi'_n$, $n = 0,1,2,\cdots$, be a normalized basis. Then the matrix of the operator Q^{-1} is symmetric in this basis. Since Q^{-1} has in the basis $\{\hat{\Xi}_n, \hat{\Xi}'_n\}$ the form

$$Q^{-1}\,\hat{\Xi}_n = c_{n+1}^{-1}c_n\,a\,q^{-2n-1}\,\hat{\Xi}_{n+1} - d_n\,\hat{\Xi}_n + c_{n-1}^{-1}c_n\,a\,q^{-2n}(1-q^n)(1-q^n/a)\,\hat{\Xi}_{n-1},$$

$$Q^{-1}\,\hat{\Xi}'_n = c'^{-1}_{n+1}c'_n\,a^{-1}q^{-2n-1}\,\hat{\Xi}'_{n+1} - d'_n\,\hat{\Xi}'_n + c'^{-1}_{n-1}c'_n\,a^{-1}q^{-2n}(1-q^n)(1-aq^n)\,\hat{\Xi}'_{n-1}.$$

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then its symmetricity means that

$$c_{n+1}^{-1} c_n \, a \, q^{-2n-1} = c_n^{-1} c_{n+1} \, a \, q^{-2n-2} (1 - q^{n+1})(1 - q^{n+1}/a),$$

$$c'^{-1}_{n+1} c'_n \, a^{-1} q^{-2n-1} = c'^{-1}_n c'_{n+1} \, a^{-1} q^{-2n-2} (1 - aq^{n+1})(1 - q^{n+1}),$$

that is,

$$\frac{c_n}{c_{n-1}} = \sqrt{\frac{q}{(1-q^n)(1-q^n/a)}}, \quad \frac{c'_n}{c'_{n-1}} = \sqrt{\frac{q}{(1-q^n)(1-aq^n)}} \, .$$

Therefore, for the coefficients c_n and c'_n we have the expressions

$$c_n = c \, \frac{q^{n/2}}{(q/a;q)_n^{1/2} (q;q)_n^{1/2}} \, , \quad c'_n = c' \, \frac{q^{n/2}}{(aq;q)_n^{1/2} (q;q)_n^{1/2}} \, ,$$

where c and c' are some constant.

Thus, in the expansions

$$\hat{\xi}_{q^n} \equiv \hat{\Xi}_n = \sum_m c_n \, \beta_m(q^n) \, |m\rangle \equiv \sum_m \hat{a}_{mn} \, |m\rangle \, ,$$

$$\hat{\xi}_{aq^n} \equiv \hat{\Xi}'_n = \sum_m c'_n \, \beta_m(aq^n) \, |m\rangle \equiv \sum_m \hat{a}'_{mn} \, |m\rangle \, ,$$

the matrix $\hat{M} := (\hat{a}_{mn} \; \hat{a}'_{mn})$ with entries

$$\hat{a}_{mn} = c_n \, \beta_m(q^n) = c \left(\frac{q^n}{(q/a;q)_n (q;q)_n)} \, \frac{q^{-m(m-1)/2}}{(q;q)_m \, (-a)^m} \right)^{1/2} U_m^{(a)}(q^n;q),$$

$$\hat{a}'_{mn} = c'_n \, \beta_m(aq^n) = c' \left(\frac{q^n}{(aq;q)_n (q;q)_n)} \, \frac{q^{-m(m-1)/2}}{(q;q)_m \, (-a)^m} \right)^{1/2} U_m^{(a)}(aq^n;q),$$

is unitary, provided that the constants c and c' are appropriately chosen. In order to calculate these constants, we use the relations $\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1$ and $\sum_{m=0}^{\infty} |\hat{a}'_{mn}|^2 = 1$ for $n = 0$. Then these sums are multiples of the sums in (8.6) and (8.7), so we find that

$$c = (a;q)_\infty^{-1/2}, \quad c' = (1/a;q)_\infty^{-1/2}. \tag{8.8}$$

The coefficients c_n and c'_n are thus real and equal to

$$c_n = \frac{q^{n/2}}{(a;q)_\infty^{1/2} (q/a;q)_n^{1/2} (q;q)_n^{1/2}}, \quad c'_n = \frac{q^{n/2}}{(1/a;q)_\infty^{1/2} (aq;q)_n^{1/2} (q;q)_n^{1/2}}.$$

The orthogonality of the matrix M means that

$$\sum_m \hat{a}_{mn} \hat{a}_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}'_{mn} \hat{a}'_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}_{mn} \hat{a}'_{mn'} = 0, \tag{8.9}$$

$$\sum_n (\hat{a}_{mn} \hat{a}_{m'n} + \hat{a}'_{mn} \hat{a}'_{m'n}) = \delta_{mm'}. \tag{8.10}$$

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Substituting into (8.10) the expressions for $\hat{a}_{mn}$ and $\hat{a}'_{mn}$, we obtain

$$\frac{1}{(a;q)_\infty}\sum_{m=0}^{\infty}\frac{q^m}{(q/a;q)_m(q;q)_m}\,U_n^{(a)}(q^m;q)\,U_{n'}^{(a)}(q^m;q)$$

$$+\frac{1}{(1/a;q)_\infty}\sum_{m=0}^{\infty}\frac{q^m}{(aq;q)_m(q;q)_m}\,U_n^{(a)}(aq^m;q)\,U_{n'}^{(a)}(aq^m;q)$$

$$=(-a)^n\,(q;q)_n\,q^{n(n-1)/2}\,\delta_{nn'}\,, \tag{8.11}$$

which must yield the orthogonality relation for Al-Salam–Carlitz I polynomials. A problem, which remains to be clarified, is the following. We have assumed that the points q^n and aq^n, $n=0,1,2,\cdots$, exhaust the whole spectrum of B_2. Let us show that this is the case.

If the operator B_2 had other spectral points x_k, then on the left-hand side of (8.11) there would be other summands $\mu_{x_k}\,U_n^{(a)}(x_k;q)\,U_{n'}^{(a)}(x_k;q)$, corresponding to these additional points. Let us show that these additional summands do not appear. For this we set $n=n'=0$ in the relation (8.11) with the additional summands. This results in the equality

$$\frac{1}{(a;q)_\infty}\sum_{m=0}^{\infty}\frac{q^m}{(q/a;q)_m(q;q)_m}+\frac{1}{(1/a;q)_\infty}\sum_{m=0}^{\infty}\frac{q^m}{(aq;q)_m(q;q)_m}+\sum_k\mu_{x_k}=1. \tag{8.12}$$

In order to show that $\sum_k\mu_{x_k}=0$, take into account the relation

$$\frac{(Aq/C,Bq/C;q)_\infty}{(q/C,ABq/C;q)_\infty}{}_2\phi_1(A,B;C;q,q)$$
$$+\frac{(A,B;q)_\infty}{(C/q,ABq/C;q)_\infty}{}_2\phi_1(Aq/C,Bq/C;q^2/C;q,q)=1$$

(see formula (2.10.13) in [21]). Putting here $A=0$, $B=0$ and $C=q/a$, we obtain relation (8.12) without the summand $\sum_k\mu_{x_k}$. Therefore, one concludes that $\sum_k\mu_{x_k}=0$. This means that additional summands do not appear in (8.11) and it does represent the orthogonality relation for the Al-Salam–Carlitz polynomials. Due to this orthogonality, we arrive at the following statement:

PROPOSITION 8.1. *The spectrum of the operator B_2 coincides with the set of points q^n and aq^n, $n=0,1,2,\cdots$. This spectrum is simple and has one accumulation point at 0.*

8.3. Duals to Al-Salam–Carlitz I polynomials. Now we consider the identities (8.9), which give the orthogonality relations for the matrix elements $\hat{a}_{mn}$ and $\hat{a}'_{mn}$, considered as functions of m. Up to multiplicative factors they coincide with

$$F_n(x;a;q)={}_2\phi_1(x,q^{-n};\;0;\;q,q^{n+1}/a)\,,\quad n=0,1,2,\cdots\,, \tag{8.13}$$

$$F'_n(x;a;q)={}_2\phi_1(x,aq^{-n};\;0;\;q,q^{n+1})\,,\quad n=0,1,2,\cdots\,, \tag{8.14}$$

considered on the set of points q^{-m}, $m=0,1,2,\cdots$. Namely, we have

$$\hat{a}_{mn}=c\left(\frac{q^n}{(q/a;q)_n(q;q)_n)}\,\frac{q^{-m(m-1)/2}}{(q;q)_m\,(-a)^m}\right)^{1/2}(-a)^m\,q^{m(m-1)/2}\,F_n(q^{-m};a;q)\,,$$

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$$\hat{a}'_{mn} = c' \left(\frac{q^n}{(aq;q)_n (q;q)_n)} \frac{q^{-m(m-1)/2}}{(q;q)_m \, (-a)^m} \right)^{1/2} (-a)^m \, q^{m(m-1)/2} \, F'_n(q^{-m};a;q) \, ,$$

where c and c' are given by (8.8). The relations (8.9) lead to the following orthogonality relations for the functions (8.13) and (8.14):

$$(a;q)^{-1}_\infty \sum_{m=0}^{\infty} \rho(m) \, F_n(q^{-m};a;q) \, F_{n'}(q^{-m};a;q) = (q/a;q)_n \, (q;q)_n \, q^{-n} \, \delta_{nn'} \, , \tag{8.15}$$

$$(1/q;q)_\infty \sum_{m=0}^{\infty} \rho(m) \, F'_n(q^{-m};a;q) \, F'_{n'}(q^{-m};a;q) = (aq;q)_n \, (q;q)_n \, q^{-n} \, \delta_{nn'} \, , \tag{8.16}$$

$$\sum_{m=0}^{\infty} \rho(m) \, F_n(q^{-m};a;q) \, F'_{n'}(q^{-m};a;q) = 0, \tag{8.17}$$

where

$$\rho(m) = \frac{(-a)^m}{(q;q)_m} \, q^{m(m-1)/2} \, .$$

Comparing the expression (8.13) for the functions $F_n(q^{-m};a;q)$ with the expression

$$C_n(q^{-m};a';q) := {}_2\phi_1(q^{-n},q^{-m};\; 0;\; q, -q^{n+1}/a')$$

for the q-Charlier polynomials, one concludes that

$$F_n(q^{-m};a;q) = C_n(q^{-m};-a;q). \tag{8.18}$$

Applying the transformation formula (see (III.6) from Appendix III in [21])

$${}_2\phi_1(q^{-n},b;\; 0;\; q,z) = (bz/q)^n \, {}_2\phi_1(q^{-n},q/z;\; 0;\; q,q/b)$$

to the expression for the functions $F'_n(q^{-m};a;q)$, we derive that

$$F'_n(q^{-m};a;q) = (-a)^{-m} \, C_n(q^{-m};-1/a;q). \tag{8.19}$$

Substituting the expressions (8.18) and (8.19) into the relations (8.15) and (8.16), we obtain the orthogonality relations for the q-Charlier polynomials $C_n(q^{-m};-a;q)$ and $C_n(q^{-m};-1/a;q)$, where $a<0$. For $C_n(q^{-m}) \equiv C_n(q^{-m};a';q)$, $a'>0$, it has the form

$$\sum_{m=0}^{\infty} \frac{a'^{\,m}}{(q;q)_m} \, q^{m(m-1)/2} \, C_n(q^{-m}) \, C_{n'}(q^{-m}) = (-a';q)_\infty \, q^{-n} (-q/a';q)_n \, (q;q)_n \, \delta_{nn'} \, .$$

It coincides with the orthogonality relation known from the literature (see, for example, Chapter 7 in [21]).

Thus, we have shown that duals of the family of Al-Salam–Carlitz I polynomials $U_n^{(a)}(q^{-m};q)$ are two sets of q-Charlier polynomials, one taken with the parameter $-a$ and the second one with the parameter $-1/a$.

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The relation (8.17) leads to the following equality for q-Charlier polynomials:

$$\sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q;q)_m} C_n(q^{-m};-a;q)\, C_{n'}(q^{-m};-1/a;q)) = 0,$$

where $a < 0$.

The set of functions (8.13) and (8.14) form an orthonormal basis in the Hilbert space $\mathfrak{l}^2$ of functions, defined on the set of points $m = 0, 1, 2, \cdots$, with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \rho(m)\, f_1(m)\, \overline{f_2(m)},$$

where $\rho(m)$ is the same as in formulas (8.15)–(8.17). One can deduce from this fact that *the q-Charlier polynomials $C_n(q^{-m};a';q)$, $a' > 0$, correspond to indeterminate moment problem and the orthogonality measure for them, obtained above, is not extremal.*

9. Duality of little q-Laguerre and Al-Salam–Carlitz II polynomials.

9.1. Pair of operators (B_3, Q^{-1}). Let $\mathcal{H} \equiv \mathcal{H}_a$ be a separable complex Hilbert space of functions, used in sections 3–7, with the polynomial basis f_n, $n = 0, 1, 2, \cdots$, in it. We fix a real number a such that $0 < a < q^{-1}$. Let B_3 be the operator on $\mathcal{H} \equiv \mathcal{H}_a$, acting upon the basis elements f_n as

$$B_3\, f_n = a_n\, f_{n+1} + a_{n-1}\, f_{n-1} + b_n\, f_n, \tag{9.1}$$

with

$$a_n = -a^{1/2} q^{n+1/2} \sqrt{(1-q^{n+1})(1-aq^{n+1})}, \quad b_n = q^n(1+a) - aq^{2k}(1+q).$$

Clearly, B_3 is a symmetric operator.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator B_3 is bounded. We assume that it is defined on the whole Hilbert space $\mathcal{H}$ and, therefore, it is a self-adjoint operator. Exactly as in the previous cases one can show that B_3 is a Hilbert–Schmidt operator. This means that the spectrum of B_3 is discrete and has a single accumulation point at 0. Moreover, a spectrum of B_3 is simple, since B_3 is representable by a Jacobi matrix with $a_n \neq 0$.

To find eigenfunctions ξ_λ of the operator B_3, $B_3\, \xi_\lambda = \lambda\, \xi_\lambda$, we set

$$\xi_\lambda = \sum_n \beta_n(\lambda)\, f_n,$$

where $\beta_n(\lambda)$ are appropriate numerical coefficients. Acting by the operator B_3 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda)\, [\, a_n\, f_{n+1} + a_{n-1}\, f_{n-1} + b_n\, f_n\,] = \lambda \sum_{n=0}^{\infty} \beta_n(\lambda)\, f_n,$$

where a_n and b_n are the same as in (9.1). Collecting in this identity all factors, which multiply f_n with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)\, a_n + \beta_{n-1}(\lambda)\, a_{n-1} + \beta_n(\lambda)\, b_n = \lambda\, \beta_n(\lambda).$$

The substitution

$$\beta_n(\lambda) = \left(\frac{(aq;q)_n}{(aq)^n (q;q)_n} \right)^{1/2} \beta'_n(\lambda)$$

reduces this relation to the following one

$$-q^n(1-aq^{n+1})\,\beta'_{n+1}(\lambda)-aq^n(1-q^n)\,\beta'_{n-1}(\lambda)+(q^n-aq^{2n+1}+aq^n-aq^{2n})\,\beta'_n(\lambda)=\lambda\,\beta'_n(\lambda).$$

This is the recurrence relation for the little q-Laguerre (Wall) polynomials

$$p_n(\lambda;a|q):={}_2\phi_1(q^{-n},0;\;aq;\;q;q\lambda)=(a^{-1}q^{-n};q)_n^{-1}\;{}_2\phi_0(q^{-n},\lambda^{-1};\;-;\;q;\lambda/a). \tag{9.2}$$

Thus, we have $\beta'_k(\lambda)=p_n(\lambda;a|q)$ and, consequently,

$$\beta_n(\lambda)=\left(\frac{(aq;q)_n}{(aq)^n(q;q)_n}\right)^{1/2}p_n(\lambda;a|q). \tag{9.3}$$

This means that eigenfunctions of the operator B_3 are of the form

$$\xi_\lambda(x)=\sum_{k=0}^{\infty}\left(\frac{(aq;q)_k}{(aq)^k(q;q)_k}\right)^{1/2}p_k(\lambda;a|q)\,f_k=\sum_{k=0}^{\infty}a^{-k/4}\,\frac{(aq;q)_k}{(q;q)_k}\,p_k(\lambda;a|q)\,x^k. \tag{9.4}$$

The expression for the eigenfunctions can be summed up. To show this one needs to know a generating function

$$F(x;\,t;\,a|q):=\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\,p_n(x;\,a|q)\,t^n \tag{9.5}$$

for the little q-Laguerre polynomials. To evaluate (9.5), we start with the second expression in (9.2) in terms of the basic hypergeometric series ${}_2\phi_0$. Substituting it into (9.5) and using the relation

$$\frac{(q^{-n};q)_k}{(q;q)_k}=(-1)^k\,q^{-kn+k(k-1)/2}\,\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

one obtains that

$$F(x;\,t;\,a|q)=\sum_{n=0}^{\infty}(-aqt)^n q^{n(n-1)/2}\sum_{k=0}^{n}\frac{(x^{-1};q)_k}{(q;q)_k(q;q)_{n-k}}(q^{-n}x/a)^k. \tag{9.6}$$

Interchanging the order of summations in (9.6) leads to the desired expression

$$F(x;\,t;\,a|q)=E_q(-aqt)\,{}_2\phi_0(x^{-1},0;\;-;\;q;xt), \tag{9.7}$$

where $E_q(z)=(-z;q)_\infty$ is the q-exponential function of Jackson.

Similarly, if one substitutes into (9.5) the explicit form of the little q-Laguerre polynomials in terms of ${}_2\phi_1$ from (9.2), this yields an expression

$$F(x;\,t;\,a|q)=\frac{E_q(-aqt)}{E_q(-t)}\,{}_2\phi_1(0,0;\;q/t;\;q;qx). \tag{9.8}$$

Using in (9.4) the explicit form of the generating function (9.7) for the little q-Laguerre polynomials, one arrives at

$$\xi_\lambda(x)=E_q(-qa^{-3/4}x)\,{}_2\phi_0(\lambda^{-1},\,0;\,-\,;\;q;\;a^{-1/4}\lambda x).$$

Another expression for $\xi_\lambda(x)$ can be written by using formula (9.8).

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Since the spectrum of the operator B_3 is discrete, only for a discrete set of values of λ the functions (9.4) belong to the Hilbert space $\mathcal{H}$. This discrete set of eigenvectors determines a spectrum of B_3.

Now we look for the spectrum of B_3 and for a set of polynomials, dual to little q-Laguerre polynomials. To this end we use the operator Q^{-1}, where Q is given by the formula $Q\, f_n = q^n\, f_n$. In order to find how the operator Q^{-1} acts upon eigenfunctions of B_3, one can use the q-difference equation

$$q^{-n}\lambda\, p_n(\lambda) = -a\, p_n(q\lambda) + (2 + a - \lambda)\, p_n(\lambda) - (1-\lambda)\, p_n(q^{-1}\lambda) \tag{9.9}$$

for the little q-Laguerre polynomials $p_n(\lambda) \equiv p_n(\lambda; a|q)$ (see formula (3.20.4) in [27]). Multiply both sides of (9.9) by $d_n\, |n\rangle$ and sum up over n, where d_n are the coefficients of $p_n(\lambda; a|q)$ in the expression (9.3) for $\beta_n(\lambda)$. Taking into account formula (9.9) and the fact that $Q^{-1}\, f_n = q^{-n}\, f_n$, one obtains the relation

$$Q^{-1}\, \xi_\lambda = -a\lambda^{-1}\, \xi_{q\lambda} + \lambda^{-1}(2 + a - \lambda)\, \xi_\lambda - \lambda^{-1}(1-\lambda)\, \xi_{q^{-1}\lambda}, \tag{9.10}$$

which is used in the next section.

9.2. The spectrum of B_3 and orthogonality of little q-Laguerre polynomials. Let us find, by using the operators B_3 and Q^{-1}, a basis in the Hilbert space $\mathcal{H}$, which consists of eigenfunctions of the operator B_3 in a normalized form, and the unitary matrix A, connecting this basis with the initial basis f_n, $n = 0, 1, 2, \cdots$, in $\mathcal{H}$. First we have to find the spectrum of B_3.

Let us first look at a form of the spectrum of B_3. If λ is a spectral point of the operator B_3, then (as it is easy to see from (9.10)) a successive action by the operator Q^{-1} upon the function (eigenfunction of B_3) ξ_λ leads to the eigenfunctions $\xi_{q^m\lambda}$, $m = 0, \pm 1, \pm 2, \cdots$. However, since B_3 is a Hilbert–Schmidt operator, not all of these points belong to the spectrum of B_3, since $q^{-m}\lambda \to \infty$ when $m \to +\infty$. This means that the coefficient ${\lambda'}^{-1} - 1$ of $\xi_{q^{-1}\lambda'}$ in (9.10) must vanish for some eigenvalue λ'. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for the coefficient of $\xi_{q^{-1}\lambda'}$ in (9.10) to vanish, that is, the point $\lambda = 1$ is a spectral point for the operator B_3. Let us show that the corresponding eigenfunction $\xi_1 \equiv \xi_{q^0}$ belongs to the Hilbert space $\mathcal{H}$.

One has the following equality

$$p_n(1; a|q) = {}_2\phi_1(q^{-n}, 0;\ aq;\ q, q) = \frac{(aq)^n}{(aq; q)_n}\, q^{n(n-1)/2}.$$

Therefore, for the scalar product $\langle \xi_1, \xi_1 \rangle$ in $\mathcal{H}$ we have

$$\langle \xi_1, \xi_1 \rangle = \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(aq)^n (q; q)_n}\, p_n^2(1; a|q) = (-1; q)_\infty.$$

Thus, the point $\lambda = 1$ does belong to the spectrum of B_3.

Let us find other spectral points of the operator B_3. Setting $\lambda = 1$ in (9.10), we see that the operator Q^{-1} transforms ξ_{q^0} into a linear combination of the vectors ξ_q and ξ_{q^0}. Moreover, ξ_q belongs to the Hilbert space $\mathcal{H}$, since the series

$$\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(aq)^n (q; q)_n}\, p_n^2(q; a|q)$$

is majorized by the corresponding series for ξ_{q^0}. Therefore, ξ_q belongs to the Hilbert space $\mathcal{H}$ and the point q is an eigenvalue of the operator B_3. Similarly, setting $\lambda = q$ in (9.10), one finds likewise that ξ_{q^2} is an eigenvector of B_3 and the point q^2 belongs to the spectrum of B_3. Repeating this procedure, we find that all ξ_{q^n}, $n=0,1,2,\cdots$, are eigenvectors of B_3 and the set q^n, $n=0,1,2,\cdots$, belongs to the spectrum of B_3. So far, we do not know yet whether other spectral points exist or not.

The functions ξ_{q^n}, $n=0,1,2,\cdots$, are linearly independent elements of the Hilbert space $\mathcal{H}$. Suppose that values q^n, $n=0,1,2,\cdots$, constitute a whole spectrum of B_3. Then the set of functions ξ_{q^n}, $n=0,1,2,\cdots$, is a basis in the Hilbert space $\mathcal{H}$. Introducing the notation $\Xi_k := \xi_{q^k}$, $k=0,1,2,\cdots$, we find from (9.10) that

$$Q^{-1}\,\Xi_k = -aq^{-k}\,\Xi_{k+1} + q^{-k}(2+a-q^k)\,\Xi_k - q^{-k}(1-q^k)\,\Xi_{k-1}. \tag{9.11}$$

As we see, the matrix of the operator Q^{-1} in the basis Ξ_k, $k=0,1,2,\cdots$, is not symmetric, although in the initial basis $|n\rangle$, $n=0,1,2,\cdots$, it was symmetric. The reason is that the matrix (a_{mn}) with entries $a_{mn} := \beta_m(q^n)$, $m,n=0,1,2,\cdots$, where $\beta_m(q^n)$ are the coefficients (9.3) in the expansion $\xi_{q^n} = \sum_m \beta_m(q^n) f_n$, is not unitary. This fact is equivalent to the statement that the basis $\Xi_n = \xi_{q^n}$, $n=0,1,2,\cdots$, is not normalized. To normalize it, one has to multiply Ξ_n by corresponding numbers c_n. Let $\hat{\Xi}_n = c_n\Xi_n$, $n=0,1,2,\cdots$, be a normalized basis. Then the matrix of the operator Q^{-1} is symmetric in this basis. It follows from (9.11) that Q^{-1} has in the basis $\{\hat{\Xi}_n\}$ the form

$$Q^{-1}\,\hat{\Xi}_n = -c_{n+1}^{-1}c_nq^{-n}a\,\hat{\Xi}_{n+1} + q^{-n}(2+a-q^n)\,\hat{\Xi}_n - c_{n-1}^{-1}c_nq^{-n}(1-q^n)\,\hat{\Xi}_{n-1}.$$

The symmetricity of Q^{-1} in the basis $\{\hat{\Xi}_n\}$ means that $c_{n+1}^{-1}c_naq^{-n} = c_n^{-1}c_{n+1}\,q^{-n-1}(1-q^{n+1})$, that is, $c_n/c_{n-1} = \sqrt{aq/(1-q^n)}$. Therefore,

$$c_n = c\,\left[(aq)^n/(q;q)_n\right]^{1/2},$$

where c is a constant.

The expansions

$$\hat{\xi}_{q^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\,\beta_m(q^n)\,f_m \equiv \sum_m \hat{a}_{mn}\,f_m$$

connect two orthonormal bases in the Hilbert space $\mathcal{H}$. This means that the matrix $(\hat{a}_{mn})$, $m,n=0,1,2,\cdots$, with entries

$$\hat{a}_{mn} = c_n\,\beta_m(q^n) = c\left(\frac{(aq)^n}{(q;q)_n}\frac{(aq;q)_m}{(aq)^m(q;q)_m}\right)^{1/2} p_m(q^n;a|q), \tag{9.12}$$

is unitary, provided that the constant c is appropriately chosen. In order to calculate this constant, we use the relation $\sum_{n=0}^{\infty}|\hat{a}_{0n}|^2 = \sum_{n=0}^{\infty}c_n^2\,\beta_0^2(q^n) = 1$. Since $\beta_0^2(q^n)=1$ and

$$\sum_{n=0}^{\infty}\frac{(aq)^n}{(q;q)_n} = (aq;q)_\infty^{-1},$$

we have

$$c = (aq;q)_\infty^{1/2}.$$

The matrix $A := (\hat{a}_{mn})$ is real and orthogonal. Thus, if $\hat{\Xi}_n$, $n=0,1,2,\cdots$, is a complete basis in $\mathcal{H}$, then $AA^{-1} = A^{-1}A = E$, that is,

$$\sum_n \hat{a}_{mn}\,\hat{a}_{m'n} = \delta_{mm'}, \qquad \sum_m \hat{a}_{mn}\,\hat{a}_{mn'} = \delta_{nn'}. \tag{9.13}$$

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Substituting into the first sum over n the expressions for $\hat{a}_{mn}$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{(aq)^n}{(q;q)_n} p_m(q^n;a|q)\, p_{m'}(q^n;aq) = \frac{(aq)^m (q;q)_m}{(aq;q)_\infty (aq;q)_m} \delta_{mm'}\,, \tag{9.14}$$

which must yield the orthogonality relation for little q-Laguerre polynomials. However, we have assumed that the points q^n, $n = 0,1,2,\cdots$, exhaust the whole spectrum of B_3. Let us show that this is the case. The reasoning here is exactly the same as in the previous sections. Namely, we have found that the spectrum of B_3 contains the points q^n, $n = 0,1,2,\cdots$. If the operator B_3 had other spectral points x_k, then on the left-hand side of (9.14) there would be other summands $\mu_{x_k}\, p_m(x_k;a|q)\, p_{m'}(x_k;a|q)$ with positive μ_{x_k}, corresponding to these additional points. Let us show that these additional summands do not appear. We set $m = m' = 0$ in the relation (9.14) with the additional summands. Since $p_0(x;a|q) = 1$, we have the equality

$$\sum_{n=0}^{\infty} \frac{(aq)^n}{(q;q)_n} + \sum_k \mu_{x_k} = (aq;q)_\infty^{-1}.$$

This formula is true only if $\sum_k \mu_{x_k} = 0$. This means that additional summands do not appear in (9.14) and thus (9.14) does represent the orthogonality relation for little q-Laguerre polynomials. Consequently, the following proposition is true:

PROPOSITION 9.1. *The spectrum of the operator B_3 coincides with the set of points q^n, $n = 0,1,2,\cdots$. This spectrum is simple and the functions ξ_{q^n}, $n = 0,1,2,\cdots$, form a complete set of eigenfunctions of B_3. The matrix $(\hat{a}_{mn})$ with entries (9.12) relates the initial basis $\{f_n\}$ with the normalized basis $\{\hat{\Xi}_n\}$.*

9.3. Al-Salam–Carlitz II polynomials as duals to little q-Laguerre polynomials. Now we consider the second relation in (9.13). Taking into account the explicit expression for $\hat{a}_{mn}$, one obtains the orthogonality relation for the functions

$$F_n(q^{-m};\, a|q) := (a^{-1}q^{-m};q)_m^{-1}\, {}_2\phi_0(q^{-m},\, q^{-n};\, -\,; q, q^n/a). \tag{9.15}$$

This relation has the form

$$\sum_{m=0}^{\infty} (aq)^{-m} \frac{(aq;q)_m}{(q;q)_m} F_n(q^{-m};\, a|q)\, F_{n'}(q^{-m};\, a|q) = (aq)^{-n} \frac{(q;q)_n}{(aq;q)_\infty} \delta_{nn'}\,. \tag{9.16}$$

Comparing (9.15) with the Al-Salam–Carlitz II polynomials

$$V_n^{(a)}(x;q) = (-a)^n q^{-n(n-1)/2} {}_2\phi_0(q^{-n},\ x;\ -\ ;\ q, q^n/a).$$

we see that they are related to the functions (9.15), and (9.16) therefore leads to the orthogonality relation for the Al-Salam–Carlitz II polynomials

$$\sum_{m=0}^{\infty} \frac{q^{m^2} a^m}{(q;q)_m (aq;q)_m} V_n^{(a)}(q^{-m};q)\, V_{n'}^{(a)}(q^{-m};q) = \frac{a^n (q;q)_n}{(aq;q)_\infty q^{n^2}} \delta_{nn'}\,,$$

known from the literature. Thus, Al-Salam–Carlitz II polynomials are duals to the little q-Laguerre polynomials.

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Appendix. In this appendix we prove the summation formula

$$\sum_{n=0}^{\infty} \frac{(abq, bq; q)_n}{(aq, q; q)_n} \frac{1-abq^{2n+1}}{1-abq} a^n q^{n^2} = \frac{(abq^2; q)_\infty}{(aq; q)_\infty}. \tag{9.17}$$

First of all, observe that when $b = 0$ this relation reduces to

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(aq, q; q)_n} = \frac{1}{(aq; q)_\infty},$$

which is a well-known limiting form of Jacobi's triple product identity (see [21], formula (1.6.3)).

One can employ an easily verified relation

$$\frac{(aq, -aq; q)_n}{(a, -a; q)_n} = \frac{1-a^2q^{2n}}{1-a^2} \tag{9.18}$$

in order to express the infinite sum in (9.17) in terms of a very-well-poised ${}_4\phi_5$ basic hypergeometric series. This results in

$$\sum_{n=0}^{\infty} \frac{(abq, bq; q)_n}{(aq, q; q)_n} \frac{1-abq^{2n+1}}{1-abq} a^n q^{n^2} = {}_4\phi_5 \left(\left. \begin{matrix} abq, \ bq, \ q\sqrt{abq}, \ -q\sqrt{abq} \\ aq, \ \sqrt{abq}, \ -\sqrt{abq}, \ 0, \ 0 \end{matrix} \right| q, aq \right).$$

The next step is to utilize a limiting case of Jackson's sum of a terminating very-well-poised balanced ${}_8\phi_7$ series,

$${}_6\phi_5 \left(\left. \begin{matrix} a, \ q\sqrt{a}, \ -q\sqrt{a}, \ b, \ c, \ d \\ \sqrt{a}, \ -\sqrt{a}, \ aq/b, \ aq/c, \ aq/d \end{matrix} \right| q, \frac{aq}{bcd} \right) = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \tag{9.19}$$

which represents a q-analogue of Dougall's formula for a very-well-poised 2-balanced ${}_7F_6$ series. When the parameters c and d tend to infinity, from (9.19) it follows that

$${}_4\phi_5 \left(\left. \begin{matrix} a, \ q\sqrt{a}, \ -q\sqrt{a}, \ b \\ \sqrt{a}, \ -\sqrt{a}, \ aq/b, \ 0, \ 0 \end{matrix} \right| q, \frac{aq}{b} \right) = \frac{(aq; q)_\infty}{(aq/b; q)_\infty}. \tag{9.20}$$

To verify this, one needs only to use the limit relation

$$\lim_{c,d\to\infty} (c, d; q)_n \left(\frac{aq}{bcd}\right)^n = q^{n(n-1)} \left(\frac{aq}{b}\right)^n.$$

With the substitutions $a \to abq$ and $b \to bq$ in (9.20), one recovers the desired identity (9.17).

Similarly, when $d \to \infty$ one derives from (9.19) the identities

$$\sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(aq, abq/c, abq; q)_n}{(1-abq)(bq, cq, q; q)_n (-a/c)^n} q^{n(n-1)/2} = \frac{(abq^2, c/a; q)_\infty}{(bq, cq; q)_\infty}, \tag{9.21}$$

$$\sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq, bq, cq; q)_n}{(1-abq)(aq, abq/c, q; q)_n (-c/a)^n} q^{n(n-1)/2} = \frac{(abq^2, a/c; q)_\infty}{(aq, abq/c; q)_\infty}. \tag{9.22}$$

They have been employed in section 7.

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We conclude this appendix with the following remark. There is another proof of the identity (4.25), based on vital use of the same summation formula (9.19). Actually, a relation may be derived, which is somewhat more general than (4.25). Indeed, consider the function

$$\eta_k(a;q) := \sum_{n=0}^{\infty}(-1)^n\, q^{n(n-1)/2}\,\frac{1-aq^{2n+1}}{1-aq}\,\frac{(aq;q)_n}{(q;q)_n}\,\mu^k(n;a)$$

for arbitrary nonnegative integers k, where the q-quadratic lattice $\mu(n;a)$ is defined as above:

$$\mu(n;a) := q^{-n} + aq^{n+1}.$$

We argue that all $\eta_k(a;q) = 0$, $k = 0, 1, 2, \cdots$. To verify this statement, begin with the case when $k = 0$ and employ relation (9.18) to show that

$$\eta_0(a;q) = {}_3\phi_3\left(\left.\begin{matrix} q\sqrt{aq},\ -q\sqrt{aq},\ aq \\ \sqrt{aq},\ -\sqrt{aq},\ 0 \end{matrix}\right| q, 1\right).$$

The summation formula (9.19) in the limit as $d \to \infty$ takes the form

$${}_5\phi_5\left(\left.\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ 0 \end{matrix}\right| q,\ \frac{aq}{bc}\right) = \frac{(aq, aq/bc;q)_\infty}{(aq/b, aq/c;q)_\infty}.$$

In the particular case when $bc = aq$ this sum reduces to

$${}_3\phi_3\left(\left.\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a} \\ 0,\ \sqrt{a},\ -\sqrt{a} \end{matrix}\right| q, 1\right) = \frac{(aq, 1;q)_\infty}{(b, c;q)_\infty} = 0\,,$$

since $(z;q)_\infty = 0$ for $z = 1$. Consequently, the function $\eta_0(a;q)$ does vanish.

For $k = 1, 2, 3, \cdots$, one can proceed inductively. Employ the relation $q\mu(n+1;a) = \mu(n;q^2a)$ to show that

$$\eta_{k+1}(a;q) = (1+aq)\eta_k(a;q) - q^{-k-1}(1-aq^2)(1-aq^3)\eta_k(aq^2;q).$$

So, one obtains that indeed all $\eta_k(a;q)$, $k = 0, 1, 2, \cdots$, vanish. The identity (4.25) is now an easy consequence of this statement if one takes into account that a product of the two polynomials $D_n(\mu(m);a,b,c|q)$ and $D_{n'}(\mu(m);b,a,abq/c|q)$ in (4.25) is some polynomial in $\mu(m)$ of degree $n+n'$. This completes the proof of (4.25), which is independent of the one, given in section 4.

REFERENCES

[1] G. E. Andrews and R. Askey, *Classical orthogonal polynomials*, in Lecture Notes in Mathematics, 1171, Springer, 1985, pp. 36–63.

[2] M. N. Atakishiyev, N. M. Atakishiyev, and A. U. Klimyk, *Big q-Laguerre and q-Meixner polynomials and representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$*, J. Phys. A: Math. Gen., 36 (2003), pp. 10335–10347.

[3] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[4] R. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, in Studies in Pure Mathematics, P. Erdös, ed., Birkhäuser, Basel, 1983, pp. 55–78.

[5] R. Askey and M. E. H. Ismail, *Recurrence relations, continued fractions and orthogonal polynomials*, Memoirs Amer. Math. Soc., 300 (1984), pp. 1–108.

[6] N. M. Atakishiyev and A. U. Klimyk, *On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials*, e-arXiv: math.CA/0307250.

[7] N. M. ATAKISHIYEV AND A. U. KLIMYK, *Hamiltonians in noncommutative world*, Contemp. Math., 389 (2005), pp. 43–67.
[8] N. M. ATAKISHIYEV AND A. U. KLIMYK, *On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials*, J. Math. Anal. Appl., 294 (2004), pp. 246–257.
[9] N. M. ATAKISHIYEV AND A. U. KLIMYK, *On discrete q-ultraspherical polynomials and their duals*, J. Math. Anal. Appl., 306 (2005), pp. 637–645.
[10] N. M. ATAKISHIYEV AND A. U. KLIMYK, *Jacobi matrix pair and dual alternative q-Charlier polynomials*, Ukrainian J. Math., 57 (2005), pp. 614–621.
[11] N. M. ATAKISHIYEV AND A. U. KLIMYK, *Representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$ and duality of q-orthogonal polynomials*, Contemp. Math., 376 (2005), pp. 195–206.
[12] N. I. AKHIEZER, *The Classical Moment Problem*, Hafner, New York, 1965.
[13] R. ASKEY, *An integral of Ramanujan and orthogonal polynomials*, J. Indian Math. Soc., 51 (1987), pp. 27–36.
[14] R. ASKEY AND J. A. WILSON, *A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols*, SIAM J. Math. Anal., 10 (1979), pp. 1008–1016.
[15] YU. M. BEREZANSKII, *Expansions in Eigenfunctions of Selfadjoint Operators*, American Mathematical Society, Providence, RI, 1968.
[16] CH. BERG AND J. P. R. CHRISTENSEN, *Density questions in the classical theory of moments*, Ann. Inst. Fourier, Grenoble, 31 (1981), pp. 99–114.
[17] CH. BERG AND M. E. H. ISMAIL, *q-Hermite polynomials and classical orthogonal polynomials*, Can. J. Math., 48 (1996), pp. 43–63.
[18] M.-P. CHEN AND H. M. SRIVASTAVA, *Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions*, Appl. Math. Comput., 68 (1995), pp. 153–188.
[19] N. CICCOLI, E. KOELINK, AND T. H. KOORNWINDER, *q-Laguerre polynomials and big q-Bessel functions and their orthogonality relations*, Meth. Appl. Anal., 6 (1999), pp. 109–127.
[20] G. K. EAGLESON, *A duality relation for discrete orthogonal systems*, Studia Scientiarum Mathematicarum Hungarica, 3 (1968), pp. 127–136.
[21] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Functions*, Cambridge University Press, Cambridge, 1990.
[22] W. GROENEVELT, *Bilinear summation formulas from quantum algebra representations*, e-arXiv: math.QA/0201272.
[23] M. E. H. ISMAIL AND D. R. MASSON, *q-Hermite polynomials, biorthogonal functions, and q-beta integrals*, Trans. Amer. Math. Soc., 346 (1994), pp. 63–116.
[24] M. E. H. ISMAIL, *Orthogonality and completeness of q-Fourier type systems*, Zeit. Anal. Anwendungen, 20 (2001), pp. 761–775.
[25] E. KOELINK AND J. KUSTERMANS, *A locally compact quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$*, Commun. Math. Phys., 233 (2003), pp. 231–296.
[26] A. KLIMYK AND K. SCHMÜDGEN, *Quantum Groups and Their Representations*, Springer, Berlin, 1997.
[27] R. KOEKOEK AND R. F. SWARTTOUW, *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and Its q-Analogue*, Delft University of Technology Report 98–17; available from ftp.tudelft.nl.
[28] D. LEONARD, *Orthogonal polynomials, duality, and associative schemes*, SIAM J. Math. Anal., 13 (1982), pp. 656–663.
[29] V. ROMANOVSKY, *Sur quelques classes nouvelles de polynomes orthogonaux*, C. R. Acad. Sci. Paris, 188 (1929), pp. 1023–1025.
[30] H. ROSENGREN, *A new quantum algebraic interpretation of the Askey–Wilson polynomials*, Contemp. Math., 254 (2000), pp. 371–394.
[31] J. SHOHAT AND J. D. TAMARKIN, *The Problem of Moments*, American Mathematical Society, Providence, New York, NY, 1943.
[32] B. SIMON, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math., 137 (1998), pp. 82–203.
[33] H. M. SRIVASTAVA AND H. L. MANOCHA, *A Treatise on Generating Functions*, Wiley, New York, 1984.
[34] F. H. SZAFRANIEC, *Duality in the quantum harmonic oscillator*, J. Phys. A: Math. Gen., 34 (2001), pp. 10487–10492.
[35] P. TERWILLIGER, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra Appl., 330 (2001), pp. 149–203.
[36] P. TERWILLIGER, *Two relations that generalize the q-Serre relations and the Dolan–Grady relations*, in Physics and Combinatorics, World Scientific, River Edge, NJ, 2001, pp. 377–398.
[37] P. TERWILLIGER, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials*, e-arXiv: math.QA/0408390.