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TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONERS FOR FOURTH-ORDER MIXED METHODS*

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Abstract. A two-level additive Schwarz preconditioning scheme for solving Ciarlet-Raviart, Hermann-Miyoshi, and Hellan-Hermann-Johnson mixed method equations for the biharmonic Dirichlet problem is presented. Using suitably defined mesh-dependent forms, a unified approach, with ties to the work of Brenner for nonconforming methods, is provided. In particular, optimal preconditioning of a Schur complement formulation for these equations is proved on polygonal domains without slits, provided the overlap between subdomains is sufficiently large.

Key words. additive Schwarz preconditioner, mixed finite elements, biharmonic equation, domain decomposition, mesh dependent norms

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction.

In this paper we introduce and analyze two-level additive Schwarz preconditioners applicable to three mixed finite element approximations for the biharmonic Dirichlet problem, valid on any bounded polygon without slits. By preconditioning the Schur complements arising from the Ciarlet-Raviart, the Herrmann-Miyoshi, and the Hellan-Herrmann-Johnson mixed methods, we obtain systems with uniformly bounded condition number when the overlap between subdomains is sufficiently large.

Additive Schwarz preconditioners for conforming [27, 29, 30], non-conforming (e.g. Morley) [4], and discontinuous Galerkin methods [14], and multigrid preconditioners for mixed finite element methods [16, 17, 24] have been developed for the biharmonic Dirichlet problem. However, the analysis of additive Schwarz preconditioners for these mixed methods is impeded by the "non-inherited" nature of the associated bilinear forms. By utilizing equivalent mesh-dependent forms, we develop a variant of the two-level additive Schwarz preconditioner proposed by Brenner [4] which applies directly to the Schur complement obtained from any of the Ciarlet-Raviart, the Herrmann-Miyoshi, or the Hellan-Herrmann-Johnson mixed methods. The equivalence of the modified Morley non-conforming and the Hellan-Hermann-Johnson mixed methods shown in [1] allows for a less direct preconditioned solution of the latter method. We note also some related work for plate problems [3, 5, 15], and second order discontinuous Galerkin methods [13, 26].

This paper is arranged in the following manner. In §2 we define a Schur complement operator for the Ciarlet-Raviart, the Herrmann-Miyoshi, and the Hellan-Herrmann-Johnson mixed methods, and prove the equivalence of a simpler mesh-dependent operator. This amounts to extending an inf-sup condition for these methods to non-convex polygons. In §3 we define a two-level additive Schwarz preconditioner and extend Brenner's analysis [4] to the case of the three aforementioned fourth-order mixed methods. In §4 we define certain intergrid operators and demonstrate that these meet the requirements of the abstract analysis of §3.

We shall let $H^s(\Omega)$ denote the L_2-based Sobolev space of order s, and denote its norm $\|\cdot\|_{s,\Omega}$, while $|\cdot|_{s,\Omega}$ will denote the associated semi-norm. Additional spaces $H_0^s(\Omega)$ are defined as the completions of $C_0^\infty(\Omega)$ with respect to the norms $\|\cdot\|_{s,\Omega}$. We shall also make reference to the negative norm (dual) spaces $H^{-s}(\Omega) = [H_0^s(\Omega)]'$. Throughout this paper we use C to denote a generic positive constant which is independent of the mesh parameter h.

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2. Background. We consider three mixed methods for the biharmonic Dirichlet problem on a polygonal domain Ω in $\mathbf{R}^2$ with boundary $\partial\Omega$:

$$\begin{aligned} \triangle^2 u = f \qquad & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \qquad & \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where $\triangle$ denotes the Laplacian operator and $\frac{\partial}{\partial\nu}$ is the normal derivative at the boundary of Ω. Each of these mixed methods is based on a weak formulation of (2.1) that can be written in the abstract form:

$$\left\{ \begin{array}{lll} \text{GIVEN:} & \text{real Banach spaces } V \text{ and } W, \text{ and } a(\cdot,\cdot), \text{ and } b(\cdot,\cdot), & \\ & \text{bilinear forms on } V\times V, \text{ and } V\times W, \text{ respectively}, & \\ \text{FIND:} & \{\sigma,u\}\in V\times W \text{ such that, for } f\in W', & \\ & \qquad a(\sigma,v) + b(v,u) = 0, & \forall\, v\in V, \\ & \qquad\qquad b(\sigma,w) = -(f,w), & \forall\, w\in W. \end{array} \right. \tag{2.2}$$

Appropriate piecewise-polynomial subspaces $V_h \subset V$ and $W_h \subset W$ are defined and an approximate solution pair $\{\sigma_h, u_h\}$ is obtained by solving:

$$\begin{array}{lll} \text{FIND:} & \{\sigma_h,u_h\}\in V_h\times W_h \text{ such that, for } f\in W', & \\ & \qquad a(\sigma_h,v) + b(v,u_h) = 0, & \forall\, v\in V_h, \\ & \qquad\qquad b(\sigma_h,w) = -(f,w), & \forall\, w\in W_h. \end{array} \tag{2.3}$$

For example, the mixed method of Ciarlet and Raviart [8] is a finite element method for the biharmonic Dirichlet problem based on a weak formulation of the form (2.2) with

$$a(\psi,\varphi) = \int_\Omega \psi\varphi\, dx\ , \qquad b(\varphi,w) = -\int_\Omega \nabla\varphi\cdot\nabla w\, dx\ . \tag{2.4}$$

The solution $\{\sigma,u\}$ of (2.2) then satisfies $\triangle^2 u = f$ and $\sigma = -\triangle u$, provided the spaces V and W are appropriately chosen. If the method is defined over a convex polygonal domain Ω, it is well-known that with $V = H^1(\Omega)$ and $W = H_0^1(\Omega)$, (2.2) yields the solution to (2.1) for each $f \in H^{-1}(\Omega)$.

Furthermore, if Ω is a (perhaps non-convex) polygon without slits, it is known [9] that, for some $\delta \in [0,\frac{1}{2})$, (2.1) has a unique solution $u \in H^{3-\delta}(\Omega)\cap H_0^2(\Omega)$ for each $f \in H^{-1-\delta}(\Omega)$. (Minimal δ values, which depend on the corner angles of $\partial\Omega$, can be estimated as in Seif [25].) When $\delta > 0$, if the forms (2.4) are interpreted as duality pairings, it is shown in [16] that, with $V = H^{1-\delta}(\Omega)$ and $W = H_0^{1+\delta}(\Omega)$, (2.2) again yields the solution of the biharmonic Dirichlet problem.

Upon constructing a regular quasiunform triangulation $\mathcal{T}_h$, with mesh diameter h, of the domain Ω, finite element subspaces for the Ciarlet-Raviart method are obtained from

$$\mathcal{S}_h^m = \{v \in C^0(\overline{\Omega}) : v|_T \in \mathcal{P}_m(T),\ \ \forall T\in\mathcal{T}_h\}\ , \tag{2.5}$$

where $\mathcal{P}_m(T)$ is the space of polynomials of degree m or less over triangle $T\in\mathcal{T}_h$. One then defines $V_h = \mathcal{S}_h^m$ and $W_h = \mathcal{S}_h^m \cap H_0^{1+\alpha}(\Omega)$. (Note that the inclusion $\mathcal{S}_h^m \subset H^{1+\alpha}(\Omega)$ can be proved by an interpolation argument for $\alpha < \frac{1}{2}$.) Throughout this paper, we shall assume that $m \geq 2$, in which case optimal order error estimates are available.

Similarly, the Hermann-Miyoshi and Hellan-Hermann-Johnson mixed methods, which solve (2.1) using the matrix of moments $\sigma_{ij} = \partial^2 u/\partial x_i\partial x_j$ as auxiliary variables, can be extended to non-convex polygons. The Hermann-Miyoshi method [2, 6, 12, 19, 20, 22] is based on the weak formulation (2.2) with

$$a(\boldsymbol{\psi},\boldsymbol{\varphi}) = \sum_{i,j=1}^{2}\int_\Omega \psi_{ij}\varphi_{ij}\, dx\ , \qquad b(\boldsymbol{\varphi},w) = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial\varphi_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i}\, dx\ ,$$

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and spaces $W = H_0^{1+\alpha}(\Omega)$, and $V = \{\boldsymbol{\varphi} = (\varphi_{ij}), 1 \le i,j \le 2 \ : \varphi_{12} = \varphi_{21},\ \varphi_{ij} \in H^{1-\alpha}(\Omega)\}$. Again, for $\alpha > 0$, the integrals in $b\,(\cdot,\cdot)$ are interpreted as duality pairings. The finite element spaces for this method are $W_h = \mathcal{S}_h^m \cap H_0^{1+\alpha}(\Omega)$, and $V_h = \{\boldsymbol{\varphi} = (\varphi_{ij}),\ 1 \le i,j \le 2\ : \varphi_{12} = \varphi_{21},\ \varphi_{ij} \in \mathcal{S}_h^m\}$.

In order to extend the Hellan-Hermann-Johnson method [2, 6, 12, 18, 19, 20, 21] to non-convex polygons it is convenient to introduce mesh-dependent spaces and norms, as in [2]. We first define on each T in $\mathcal{T}_h$

$$M_\nu(\boldsymbol{\varphi}) = \sum_{i,j=1}^{2} \varphi_{ij}\nu_j\nu_i \ \text{ and } \ M_{\nu\tau}(\boldsymbol{\varphi}) = \sum_{i,j=1}^{2} \varphi_{ij}\nu_j\tau_i \ ,$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to ∂T, and $\tau = (\tau_1, \tau_2) = (\nu_2, -\nu_1)$, is the unit tangent vector at the boundary ∂T. Then with

$$H_h^2 = \{u \in H^1(\Omega) :\ u|_T \in H^2(T),\ \ \forall T \in \mathcal{T}_h\}$$

and

$$\begin{aligned} \overset{\circ}{\mathcal{V}}_h(\Omega) = \{\boldsymbol{\varphi} = (\varphi_{ij}),\ 1 \le i,j \le 2 :\ \varphi_{12} = \varphi_{21},\ \ \varphi_{ij}|_T \in H^{1-\alpha}(T),\ \forall T \in \mathcal{T}_h \\ \text{and } M_\nu(\boldsymbol{\varphi}) \text{ is continuous at the interelement boundaries}\} \end{aligned}$$

we define $W = H_h^2 \cap H_0^{1+\alpha}(\Omega)$, while V is taken to be the completion of $\overset{\circ}{\mathcal{V}}_h(\Omega)$ with respect to the norm

$$\|\boldsymbol{\varphi}\|_{0,h} = \sqrt{\sum_{i,j} \int_\Omega |\varphi_{ij}|^2\ dx + h \int_{\Gamma_h} |M_\nu(\boldsymbol{\varphi})|^2\ ds}\ ,$$

where $\Gamma_h = \cup_{T\in\mathcal{T}_h}\partial T$. We denote the common mesh-dependent norm for H_h^2

$$\|\!|w|\!\|_{2,h} = \sqrt{\sum_{T\in\mathcal{T}_h} \|w\|_{2,T}^2 + h^{-1}\int_{\Gamma_h} \left|\mathcal{J}\frac{\partial w}{\partial \nu}\right|^2\ ds}\ , \tag{2.6}$$

where, if e is an interior edge of $\mathcal{T}_h$, $\mathcal{J}\frac{\partial w}{\partial\nu}\Big|_e$ denotes the (signed) jump in the (outward) normal derivative of w at a point on e, and for boundary edges, we first extend w by zero (cf. [2]). With

$$\begin{aligned} a\,(\boldsymbol{\psi},\boldsymbol{\varphi}) &= \sum_{i,j=1}^{2} \int_\Omega \psi_{ij}\varphi_{ij}\,dx\ , \\ b\,(\boldsymbol{\varphi},w) &= \sum_{T\in\mathcal{T}_h} \left\{ -\sum_{i,j=1}^{2}\int_T \varphi_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j}\,dx + \int_{\partial T} M_\nu(\boldsymbol{\varphi})\frac{\partial w}{\partial\nu}\,ds \right\} \end{aligned}$$

it can be shown that (2.2) has a unique solution pair $\{\boldsymbol{\sigma}, u\}$ with u satisfying (2.1) and $\sigma_{ij} = \partial^2 u/\partial x_i \partial x_j$.

Finite element spaces for the Hellan-Hermann-Johnson method are

$$V_h = \{\boldsymbol{\varphi} \in V : \varphi_{ij}|_T \in \mathcal{P}_{m-1}(T),\ \forall T \in \mathcal{T}_h\}\ ,$$

and $W_h = \mathcal{S}_h^m \cap H_0^{1+\alpha}(\Omega)$. We also note that as a result of the Green's formula

$$
\begin{aligned}
&-\sum_{i,j=1}^{2}\int_T \varphi_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j}\,dx + \int_{\partial T} M_\nu(\boldsymbol{\varphi})\frac{\partial w}{\partial \nu}\,ds \\
&\qquad = \sum_{i,j=1}^{2}\int_T \frac{\partial \varphi_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i}\,dx - \int_{\partial T} M_{\nu\tau}(\boldsymbol{\varphi})\frac{\partial w}{\partial \tau}\,ds\;, \qquad (2.7)
\end{aligned}
$$

valid for $\boldsymbol{\varphi} \in H^1(T)$ and $w \in H^2(T)$, one may reexpress the bilinear form $b(\cdot,\cdot)$. Furthermore, the Hellan-Hermann-Johnson forms reduce to those of the Hermann-Miyoshi method when restricted to the latter's finite element spaces (for which $\sum_{T\in\mathcal{T}_h}\int_{\partial T} M_{\nu\tau}(\boldsymbol{\varphi})\frac{\partial w}{\partial \tau}\,ds = 0$). Consequently, for both methods the Schwarz inequality yields

$$
b(\boldsymbol{\varphi}, w) \le \|\boldsymbol{\varphi}\|_{0,h}\|w\|_{2,h} \qquad \forall\, \boldsymbol{\varphi} \in V_h, w \in W_h\;, \qquad (2.8)
$$

where

$$
\|w\|_{2,h} = \sqrt{\sum_{T\in\mathcal{T}_h} |w|_{2,T}^2 + h^{-1}\int_{\Gamma_h}\left|\boldsymbol{\mathcal{J}}\frac{\partial w}{\partial \nu}\right|^2\,ds}\;, \qquad (2.9)
$$

and $|\!|\!|\cdot|\!|\!|_{2,h}^2 = \|\cdot\|_{2,h}^2 + \|\cdot\|_1^2$.

Choosing bases $\{\varphi^i\}$ for V_h and $\{\phi^j\}$ for W_h, consider the linear system (2.3) in block matrix form with the notations $[\mathcal{B}_h]_{ij} = b(\varphi^j, \phi^i)$, $[\mathcal{A}_h]_{ij} = (\varphi^i, \varphi^j)_{L_2}$, $[\boldsymbol{f}]_j = (f, \phi^j)$, and denote the transpose of $\mathcal{B}_h$ by $\mathcal{B}_h^t$:

$$
N\begin{pmatrix} \boldsymbol{\sigma_h} \\ \boldsymbol{u_h} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_h & \mathcal{B}_h^t \\ \mathcal{B}_h & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma_h} \\ \boldsymbol{u_h} \end{pmatrix} = \begin{pmatrix} 0 \\ -\boldsymbol{f} \end{pmatrix}. \qquad (2.10)
$$

Applying block Gaussian elimination one obtains the reduced system for (the coefficients of) $\boldsymbol{u_h}$

$$
\mathcal{B}_h\mathcal{A}_h^{-1}\mathcal{B}_h^t\,\boldsymbol{u_h} = \boldsymbol{f}\,. \qquad (2.11)
$$

In principle, the action of the L_2-Gram matrix $\mathcal{A}_h^{-1}$ may be computed with a rapidly converging iteration; although, a method is described in [17] for avoiding this iteration. With $\boldsymbol{u_h}$ obtained from (2.11), one may then compute $\boldsymbol{\sigma_h} = -\mathcal{A}_h^{-1}\mathcal{B}_h^t\boldsymbol{u_h}$.

Unlike the indefinite block matrix N, the similarly ill-conditioned Schur complement $\mathcal{B}_h\mathcal{A}_h^{-1}\mathcal{B}_h^t$ is symmetric positive definite. One approach to solving (2.3) is to construct a preconditioner for $\mathcal{B}_h\mathcal{A}_h^{-1}\mathcal{B}_h^t$. To this end we note that $\mathcal{B}_h\mathcal{A}_h^{-1}\mathcal{B}_h^t$ induces an inner product (bilinear form) $S_h(v,w) =< \mathcal{B}_h\mathcal{A}_h^{-1}\mathcal{B}_h^t v, w >$ on W_h, where $< \cdot,\cdot >$ denotes the Euclidean inner product. It is not difficult to show (cf. (3.19) in [17]) that,

$$
S_h(w,w)^{1/2} = \sup_{v\in V_h\setminus\{0\}} \frac{b(v,w)}{\sqrt{a(v,v)}}\,. \qquad (2.12)
$$

Rather than constructing a preconditioner for this Schur complement directly, we introduce an auxiliary mesh-dependent bilinear form

$$
\begin{aligned}
a_h(v,w) = &\left(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{2}\int_T \partial_{ij}^2 v\,\partial_{ij}^2 w\;dx\right. \\
&\left. + h^{-1}\int_{\Gamma_h}\left(\boldsymbol{\mathcal{J}}\frac{\partial v}{\partial \nu}\right)\left(\boldsymbol{\mathcal{J}}\frac{\partial w}{\partial \nu}\right)\,ds\right. \qquad \forall v,w \in W_h\;, \qquad (2.13)
\end{aligned}
$$

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and operators $A_h : W_h \to W_h$, and $S_h : W_h \to W_h$ defined by

$$(A_h v, w)_h = a_h(v, w) \qquad \forall\, v, w \in W_h \ , \tag{2.14}$$

$$(S_h v, w)_h = S_h(v, w) \qquad \forall\, v, w \in W_h \ . \tag{2.15}$$

We also note that,

$$|\!|\!| w |\!|\!|_{2,h}^2 = a_h(w, w), \tag{2.16}$$

and by virtue of the quasiuniformity of $\mathcal{T}_h$, standard inverse properties, and Lemma 2 from [2]

$$|\!|\!| w |\!|\!|_{2,h} \le C\, h^{-2} \|w\|_0 \qquad \forall\, w \in \mathcal{S}_h^m \ . \tag{2.17}$$

In the next Theorem we prove that A_h and S_h are spectrally equivalent, and hence a preconditioner for A_h, such as the one described in §3, also preconditions the Schur complement operator S_h. By virtue of (2.12), proving this spectral equivalence will be tantamount to proving one of the so-called inf-sup conditions (in terms of the mesh-dependent norms $\|\cdot\|_{0,h}$ and $|\!|\!|\cdot|\!|\!|_{2,h}$). Such a proof is given for convex Ω in [2]. We shall generalize this proof to possibly non-convex polygons whose triangulation $\mathcal{T}_h$ can be extended to a regular and quasiuniform triangulation $\tilde{\mathcal{T}}_h$ of a larger *convex* polygonal domain $\tilde{\Omega} \supset \Omega$ (while retaining the same mesh diameters). The following result concerning the norm

$$\|\varphi\|_{0,h} = \sqrt{\int_\Omega |\varphi|^2 \, dx + h \int_{\Gamma_h} |\varphi|^2 \, ds}\ , \tag{2.18}$$

(see the proof of Lemma 1 in [2]), will also be needed

LEMMA 2.1. *There is a constant C such that,*

$$\|\varphi\|_{0,h} \le C\, \|\varphi\|_0\ , \tag{2.19}$$

for all φ such that , $\varphi|_T \in \mathcal{P}_m(T), \quad \forall T \in \mathcal{T}_h$.

Consequently, $\|\varphi\|_{0,h} \le C\, \sqrt{a\,(\varphi, \varphi)}$, and the same result holds for Hermann-Miyoshi and Hellan-Hermann-Johnson elements $\boldsymbol{\varphi} = (\varphi_{ij})$ since $\|\boldsymbol{\varphi}\|_{0,h}^2 \le \sum_{i,j} \|\varphi_{ij}\|_{0,h}^2$. We define

$$\tilde{\mathcal{S}}_h^m = \left\{ v \in C^0(\overline{\Omega}) : v|_T \in \mathcal{P}_m(T) \ \ \forall T \in \tilde{\mathcal{T}}_h, \ \text{ and } \ \int_{\tilde{\Omega}} v \, dx = 0 \right\} \ .$$

THEOREM 2.2. *If the triangulation $\mathcal{T}_h$ can be extended to $\tilde{\mathcal{T}}_h$ as described above, then the Schur complement form $S_h(\cdot,\cdot)$, and the mesh-dependent form $a_h(\cdot,\cdot)$ are spectrally equivalent, i.e. there exist constants C_1 and C_2, independent of h, such that,*

$$C_1\, a_h(w, w) \le S_h(w, w) \le C_2\, a_h(w, w) \qquad \forall\, w \in W_h\ . \tag{2.20}$$

Proof. First, by the Schwarz inequality

$$|b\,(\varphi, w)| \le C\, \|\varphi\|_{0,h} |\!|\!| w |\!|\!|_{2,h} \tag{2.21}$$

where $\|\cdot\|_{0,h}$ is the choice appropriate for the given method, cf. (2.8). Then since $\|\varphi\|_{0,h} \le C\, \sqrt{a\,(\varphi, \varphi)}$ follows from Lemma 2.1 as noted above, dividing by $\|\varphi\|_{0,h}$ and using (2.12) yields the righthand inequality in (2.20). It follows that $|\!|\!|\cdot|\!|\!|_{2,h}$ is a norm on W_h.

For the lefthand inequality, consider first the Ciarlet-Raviart method. For each $w \in W_h$ we extend by zero from Ω to $\tilde{\Omega}$, and setting $e = \int_\Omega w \, dx \, / \, |\tilde{\Omega}|$, we define $\tilde{w} = w - e \in \tilde{\mathcal{S}}_h^m$. Then there exists $\tilde{v} \in \tilde{\mathcal{S}}_h^m$ such that,

$$\int_{\tilde{\Omega}} \tilde{v}\xi \, dx = \int_{\tilde{\Omega}} \nabla \tilde{w} \cdot \nabla \xi \, dx \qquad \forall \, \xi \in \tilde{\mathcal{S}}_h^m \, . \tag{2.22}$$

Note that, with $v = \tilde{v}|_\Omega$

$$\|v\|_{0,\Omega} \leq \|v\|_{0,h,\Omega} \leq \|\tilde{v}\|_{0,h,\tilde{\Omega}} \, . \tag{2.23}$$

Setting $\xi = \tilde{v}$ in (2.22) and using Lemma 2.1,

$$C \, \|\tilde{v}\|_{0,h,\tilde{\Omega}}^2 \leq \int_{\tilde{\Omega}} \tilde{v}^2 \, dx = \int_{\tilde{\Omega}} \nabla \tilde{w} \cdot \nabla \tilde{v} \, dx \leq |b\,(v,w)| \, . \tag{2.24}$$

Using estimates for the Neumann projection on $\tilde{\Omega}$ and its piece-wise linear interpolant, along with the regularity available on the convex $\tilde{\Omega}$, it is shown in Lemma 5 of [2] that,

$$|||\tilde{w}|||_{2,h,\tilde{\Omega}} \leq C \, \|\tilde{v}\|_{0,h,\tilde{\Omega}} \, . \tag{2.25}$$

Then since $\|w\|_{2,h,\Omega}^2 = \|\tilde{w}\|_{2,h,\tilde{\Omega}}^2 \leq |||\tilde{w}|||_{2,h,\tilde{\Omega}}^2$, combining (2.23), (2.24), and (2.25) yields

$$C \, \|v\|_{0,\Omega} \|w\|_{2,h,\Omega} \leq C \, \|\tilde{v}\|_{0,h,\tilde{\Omega}}^2 \leq \int_\Omega \nabla w \cdot \nabla v \, dx \leq |b\,(v,w)| \, . \tag{2.26}$$

By (2.16) and (2.12), the lefthand inequality in (2.20) follows for the Ciarlet-Raviart method after dividing (2.26) by $\|v\|_{0,\Omega} = \sqrt{a\,(v,v)}$.

In the case of the Hermann-Miyoshi and Hellan-Hermann-Johnson methods, for each $w \in W_h$ we first associate $v \in \mathcal{S}_h^m$ satisfying (2.26), then define $\mathbf{v} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$. For this choice of $\mathbf{v}$, it follows from the Green's formula (2.7) that $b\,(\mathbf{v}, w)$ reduces to $\int_\Omega \nabla w \cdot \nabla v \, dx$ in both cases. Furthermore, $\sqrt{a\,(\mathbf{v},\mathbf{v})} = \sqrt{\sum_{i,j} \|v_{ij}\|_0^2} \leq \sqrt{2}\|v\|_0$, and the proof is completed again by combining (2.26) and (2.12). $\square$

3. Two-level analysis. In this section we describe a two-level additive Schwarz preconditioner, B, for A_h, and hence also for the spectrally equivalent Schur complement operators obtained for any of the three mixed methods described in §2. We then show that the condition number of BA_h is bounded independent of mesh sizes and the number of subdomains, provided there is sufficient overlap (cf. Theorem 3.1).

We shall assume that $\Omega = \cup_{j=1}^J \Omega_j$ is a partitioning into overlapping open subdomains and that $\mathcal{T}_H$ is a quasiuniform triangulation of Ω for which $\mathcal{T}_h$ is a finer subdivision that is also aligned with each Ω_j. We further assume the existence (see [10]) of a C^∞ partition of unity, $\theta_1, \theta_2, \ldots, \theta_J$ satisfying

$$\theta_j = 0 \qquad \text{on } \Omega \setminus \Omega_j \, , \tag{3.1}$$

$$\sum_{j=1}^J \theta_j = 1 \qquad \text{on } \overline{\Omega} \, , \tag{3.2}$$

$$\|\nabla \theta_j\|_\infty \leq \frac{C}{\delta} \, , \qquad \|\nabla^2 \theta_j\|_\infty \leq \frac{C}{\delta^2} \, , \tag{3.3}$$

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where $0 < h \le C_1\,\delta$ and $0 < \delta \le C_2\,H$. Let W_j be the subspace of W_h whose members vanish except at nodes interior to Ω_j, and let W_H, (equipped with an inner product $(\cdot,\cdot)_H$) denote the finite element space associated with $\mathcal{T}_H$ and taking zero values at boundary nodes. Furthermore, we denote by $\overline{N}_c$ the maximum number of overlaps for the *closures* of the subdomains Ω_j, which we assume to be independent of h, H, δ, and J.

We define $\|w\|_{2,H}$ and $a_H(v,w)$ in analogy with (2.9) and (2.13). With $\mathcal{T}_{h,j} = \{T \,|\, T \in \mathcal{T}_h, T \subset \Omega_j\}$ and $\Gamma_j = \bigcup_{T\in\mathcal{T}_{h,j}} \partial T$ we also define

$$\|w\|_{2,j} = \sqrt{\sum_{T\in\mathcal{T}_{h,j}} |w|^2_{2,T} + h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial w}{\partial \nu}\right|^2\,ds}\;. \tag{3.4}$$

In order to construct the preconditioner B we first define four discrete operators $A_H : W_H \to W_H$, $A_j : W_j \to W_j$, $Q_j : W_h \to W_j$, and $P_j : W_h \to W_j$ by

$$(A_H v, w)_H = a_H(v,w) \qquad \forall\, v,w \in W_H\;, \tag{3.5}$$
$$(A_j v, w)_h = a_h(v,w) \qquad \forall\, v,w \in W_j\;, \tag{3.6}$$
$$(Q_j v, w)_h = (v,w)_h \qquad \forall\, v \in W_h, w \in W_j\;, \tag{3.7}$$
$$a_h(P_j v, w) = a_h(v,w) \qquad \forall\, v \in W_h, w \in W_j\;. \tag{3.8}$$

We shall also construct an intergrid transfer operator $I_H^h : W_H \to W_h$, and will consider the two adjoints $I_h^H : W_h \to W_H$ and $P_h^H : W_h \to W_H$ defined by

$$(I_h^H v, w)_H = (v, I_H^h w)_h \qquad \forall\, v \in W_h, w \in W_H\;, \tag{3.9}$$
$$a_H(P_h^H v, w) = a_h(v, I_H^h w) \qquad \forall\, v \in W_h, w \in W_H\;. \tag{3.10}$$

The operator I_h^H makes use of an intermediate mapping to quintic Argyris elements and is defined in §4.

Additionally, we require positive definite operators R_H and R_j, symmetric with respect to $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_h$, respectively, which are approximate solvers for A_H and A_j. We denote

$$\omega_0 = \min\left(\lambda_{\min}(R_H A_H), \lambda_{\min}(R_1 A_1), \ldots, \lambda_{\min}(R_J A_J)\right)\;, \tag{3.11}$$
$$\omega_1 = \max\left(\lambda_{\max}(R_H A_H), \lambda_{\max}(R_1 A_1), \ldots, \lambda_{\max}(R_J A_J)\right)\;, \tag{3.12}$$

and observe that if, for example, R_H and R_j are chosen to be appropriate multigrid preconditioners for the corresponding Schur complements, then ω_1/ω_0 is bounded independent of h, H, δ, and J, (cf. [17]).

The two-level additive Schwarz preconditioner B is then defined as in [4] by

$$B = I_H^h R_H I_h^H + \sum_{j=1}^{J} R_j Q_j\;. \tag{3.13}$$

It is not difficult to show that $A_H P_h^H = I_h^H A_h$ and $A_j P_j = Q_j A_h$ so that,

$$\begin{aligned} BA_h &= I_H^h R_H I_h^H A_h + \sum_{j=1}^{J} R_j Q_j A_h \\ &= I_H^h R_H A_H P_h^H + \sum_{j=1}^{J} R_j A_j P_j\;. \end{aligned} \tag{3.14}$$

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As in [4] we rely on four pairs of assumptions to bound the condition number of BA_h. The first two of these are:

A.1a $\quad \left\| I_H^h w \right\|_{2,h} \le C \|w\|_{2,H} \qquad \forall\, w \in W_H$,

A.1b $\quad \left| I_H^h w - w \right|_l \le C H^{2-l} \|w\|_{2,H} \qquad \forall\, w \in W_H, \quad 0 \le l \le 1$,

A.2a $\quad \sqrt{a_h(w,w)}$ (resp. $\sqrt{a_H(w,w)}$) is equivalent to $\|w\|_{2,h}$ (resp. $\|w\|_{2,H}$) $\forall\, w \in W_h$ (resp. $w \in W_H$),

A.2b $\quad a_h(v,w) \le C \|v\|_{2,j} \|w\|_{2,h} \qquad \forall\, v \in W_h, w \in W_j$.

Furthermore, we posit the existence of an additional intergrid transfer operator $J_h^H : W_h \to W_H$ satisfying

A.3a $\quad \left\| J_h^H w \right\|_{2,H} \le C \|w\|_{2,h} \qquad \forall\, w \in W_h$,

A.3b $\quad \left| J_h^H w - w \right|_l \le C H^{2-l} \|w\|_{2,h} \qquad \forall\, w \in W_h, \quad 0 \le l \le 1$,

and

A.4a $\quad \|\Pi_h(\lambda w)\|_{2,j} \le C \|\lambda w\|_{2,j} \qquad \forall\, w \in \mathcal{S}_h^m, \lambda \in \mathcal{S}_h^1$,

A.4b $\quad \|\Pi_h(gv)\|_{0,T} \le C \|g\|_\infty \|v\|_{0,T} \qquad \forall T \in \mathcal{T}_h, v \in \mathcal{P}_m(T), g \in C^\infty(\overline{T})$,

where Π_h is the nodal interpolation operator associated with $\mathcal{S}_h^m$, and C depends only on the minimum angle in $\mathcal{T}_h$.

For assumption A.2a we actually have an equality, (2.16). The proof of A.2b follows from the Schwarz inequality applied to the integral formulation (2.13) for $a_h(v,w)$, on accounting for the support of $w \in W_j$. Note that $\|w\|_{2,j}$ depends on $v|_T$ also for triangles T *adjacent* to Ω_j through the term $h^{-1} \int_{\Gamma_j} \left| \mathcal{J} \frac{\partial v}{\partial \nu} \right|^2 \, ds$. Assumptions A.1 and A.3 are proved in the next section. Assumption A.4a is a consequence of the equivalence of the discrete L_2-norm, $C_1 \|v\|_{0,T}^2 \le |T| \sum_{n_i} v(n_i)^2 \le C_2 \|v\|_{0,T}^2$. Finally, Assumption A.4b is proved using inverse properties, a trace inequality, and a standard scaling argument like that used for Lemma 4 in [2]. With these assumptions, one may prove the following:

THEOREM 3.1. *Given a partition of unity satisfying (3.1)–(3.3), and assuming* A.1 − 4 *we have*

$$\frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C \, \frac{\omega_1}{\omega_0} \, \overline{N}_c^{\;2} \left(1 + \left(\frac{H}{\delta} \right)^4 \right) .$$

Proof. We first estimate $\lambda_{\max}(BA_h)$. By virtue of (3.14), (3.10), and the properties of $R_H A_H$ and $R_j A_j$

$$\begin{aligned} a_h(BA_h w, w) &= a_H(R_H A_H P_h^H w, P_h^H w) + \sum_{j=1}^J a_h(R_j A_j P_j w, P_j w) \\ &\le \omega_1 \left[a_H(P_h^H w, P_h^H w) + \sum_{j=1}^J a_h(P_j w, P_j w) \right] . \end{aligned} \tag{3.15}$$

By (3.10), the Schwarz inequality, (A.2a), and (A.1a)

$$\begin{aligned} a_H(P_h^H w, P_h^H w) &= a_h(w, I_H^h P_h^H w) \\ &\le a_h(w,w)^{1/2} a_h(I_H^h P_h^H w, I_H^h P_h^H w)^{1/2} \\ &\le a_h(w,w)^{1/2} C \left\| P_h^H w \right\|_{2,H} \\ &\le C \, a_h(w,w)^{1/2} a_H(P_h^H w, P_h^H w)^{1/2} \\ &\le C^2 a_h(w,w) . \end{aligned} \tag{3.16}$$

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Furthermore, by (3.8) and (A.2)

$$a_h(P_jw, P_jw) = a_h(w, P_jw) \tag{3.17}$$
$$\le \|w\|_{2,j}\|P_jw\|_{2,h}$$
$$\le C\,\|w\|_{2,j}\sqrt{a_h(P_jw, P_jw)}$$
$$\le C^2\|w\|_{2,j}^2$$

Summing (3.17) over j, then combining with (3.15), (3.16), and since $\sum_{j=1}^{J}\|w\|_{2,j}^2 \le \overline{N}_c\|w\|_{2,h}^2$ for $w \in W_h$, we find

$$\lambda_{\max}(BA_h) \le C\,\omega_1\overline{N}_c\ .$$

It remains to show that,

$$\lambda_{\min}(BA_h) \ge \frac{C\,\omega_0}{\overline{N}_c\,\left(1+\left(\frac{H}{\delta}\right)^4\right)}\ .$$

This is a consequence of Lemmas 2.3 and 2.4 in [4] which in turn employ the ideas of Dryja and Widlund [10, 11], (see also [23, 28]). The proof of Lemma 2.4 needs no change here. However, the interelement boundary integral term in $\|w\|_{2,h}$ requires one provide a slightly different argument than that found in Brenner's Lemma 2.3. This argument, which completes the proof of Theorem 3.1, is provided in the following Lemma 3.2. ∎

LEMMA 3.2. *Given any $v \in W_h$, there exist $v_0 \in W_H$ and $v_j \in W_j$ ($1 \le j \le J$) such that,*

$$v = I_H^h\,v_0 + \sum_{j=1}^{J} v_j\ , \tag{3.18}$$

and

$$a_H(v_0, v_0) + \sum_{j=1}^{J} a_h(v_j, v_j) \le C\,\overline{N}_c\left(1+\left(\frac{H}{\delta}\right)^4\right)a_h(v,v)\ . \tag{3.19}$$

Proof. As in [4] we let $v_0 = J_h^H v$, $w = v - I_H^h v_0$, and $v_j = \Pi_h(\theta_j w)$, where Π_h is the nodal interpolation operator of Assumption A.4. Then (3.18) holds. The following are simple consequences of A.1 − 3 and the triangle inequality, (cf. (2.31) in [4])

$$a_H(v_0, v_0) \le C\,a_h(v,v), \tag{3.20}$$
$$a_h(w,w) \le C\,a_h(v,v), \qquad \text{and} \tag{3.21}$$
$$\|w\|_0 + H\,|w|_1 \le C\,H^2\sqrt{a_h(v,v)}\ . \tag{3.22}$$

It remains to show that,

$$a_h(v_j, v_j) \le C\left(\|w\|_{2,j}^2 + \delta^{-2}|w|_{1,\Omega_j}^2 + \delta^{-4}\|w\|_{0,\Omega_j}^2\right)\ , \tag{3.23}$$

and hence

$$\sum_{j=1}^{J} a_h(v_j, v_j) \le C\,\overline{N}_c\,\left(a_h(w,w) + \delta^{-2}|w|_1^2 + \delta^{-4}\|w\|_0^2\right)\ . \tag{3.24}$$

Then (3.19) follows on combining (3.20)–(3.24).

We shall employ the piecewise-linear interpolant $\tilde{\theta_j}$ of θ_j, which satisfies on each triangle T

$$\left\|\tilde{\theta_j}\right\|_{\infty,T} \leq \|\theta_j\|_{\infty,T}\,, \qquad \left\|\nabla\tilde{\theta_j}\right\|_{\infty,T} \leq C\,\|\nabla\theta_j\|_{\infty,T}\,, \tag{3.25}$$

and

$$\left\|\theta_j - \tilde{\theta_j}\right\|_{\infty,T} + h\left\|\nabla(\theta_j - \tilde{\theta_j})\right\|_{\infty,T} \leq C\,h^2\left\|\nabla^2\theta_j\right\|_{\infty,T}\,, \tag{3.26}$$

where C depends only on the minimum angle of $\mathcal{T}_h$. Combining the Schwarz inequality, the inverse property (2.17) for $\|\cdot\|_{2,h}$, A.4, (3.25), (3.26), and (3.3), one obtains

$$\begin{aligned}
\|v_j\|_{2,j}^2 &= \|\Pi_h(\theta_j w)\|_{2,j}^2 \\
&= \left\|\Pi_h(\tilde{\theta_j} w) + \Pi_h((\theta_j - \tilde{\theta_j})w)\right\|_{2,j}^2 \\
&\leq 2\,\left(\left\|\Pi_h(\tilde{\theta_j} w)\right\|_{2,j}^2 + \left\|\Pi_h((\theta_j - \tilde{\theta_j})w)\right\|_{2,j}^2\right) \\
&\leq C\,\left(\left\|\tilde{\theta_j} w\right\|_{2,j}^2 + h^{-4}\left\|\Pi_h((\theta_j - \tilde{\theta_j})w)\right\|_{0,\Omega_j}^2\right) \\
&\leq C\,h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} w\right)\right|^2\,ds + C\sum_{T\in\mathcal{T}_{h,j}}\left|\tilde{\theta_j} w\right|_{2,T}^2 \\
&\qquad\qquad + C\,h^{-4}\left\|\Pi_h((\theta_j - \tilde{\theta_j})w)\right\|_{0,\Omega_j}^2 \\
&\leq C\,h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} w\right)\right|^2\,ds \\
&\qquad\qquad + C\sum_{T\in\mathcal{T}_{h,j}}\left(\left\|\tilde{\theta_j}\right\|_{\infty,T}|w|_{2,T} + \left\|\nabla\tilde{\theta_j}\right\|_{\infty,T}|w|_{1,T}\right)^2 \\
&\qquad\qquad + C\,h^{-4}\left\|\theta_j - \tilde{\theta_j}\right\|_{\infty,\Omega_j}^2\|w\|_{0,\Omega_j}^2) \\
&\leq C\,h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} w\right)\right|^2\,ds \\
&\qquad\qquad + C\sum_{T\in\mathcal{T}_{h,j}}|w|_{2,T}^2 + \delta^{-2}\,|w|_{1,\Omega_j}^2 + \delta^{-4}\,\|w\|_{0,\Omega_j}^2\,.
\end{aligned} \tag{3.27}$$

In order to establish (3.23) and hence (3.24) we must show that,

$$h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} w\right)\right|^2\,ds \tag{3.28}$$

is appropriately bounded. To this end note that by the product rule, the Schwarz inequality, and since $\mathcal{J}\frac{\partial\theta_j}{\partial\nu} \equiv 0$ on Γ_j,

$$\begin{aligned}
\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} w\right)\right|^2 &\leq 2\left|\tilde{\theta_j}\,\mathcal{J}\frac{\partial w}{\partial\nu}\right|^2 + 2\left|w\,\mathcal{J}\frac{\partial\tilde{\theta_j}}{\partial\nu}\right|^2 \\
&\leq 2\left|\tilde{\theta_j}\,\mathcal{J}\frac{\partial w}{\partial\nu}\right|^2 + 2\left|w\,\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta_j} - \theta_j\right)\right|^2\,.
\end{aligned} \tag{3.29}$$

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Hence, by (3.29), (3.26), (3.25), (3.3), and Lemma 2.1,

$$\begin{aligned} h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial}{\partial\nu}\left(\tilde{\theta}_j w\right)\right|^2\,ds \\ &\le C\,h^{-1}\left(\left\|\tilde{\theta}_j\right\|^2_{\infty,\overline{\Omega}_j}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial w}{\partial\nu}\right|^2\,ds+\left\|\nabla(\theta_j-\tilde{\theta}_j)\right\|^2_{\infty,\overline{\Omega}_j}\int_{\Gamma_j}|w|^2\,ds\right) \\ &\le C\,h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial w}{\partial\nu}\right|^2\,ds+C\,h\left\|\nabla^2\theta_j\right\|^2_{\infty,\overline{\Omega}_j}\int_{\Gamma_j}|w|^2\,ds \\ &\le C\,h^{-1}\int_{\Gamma_j}\left|\mathcal{J}\frac{\partial w}{\partial\nu}\right|^2\,ds+C\,\delta^{-4}\|w\|^2_{0,\Omega_j} \end{aligned} \tag{3.30}$$

Combining (3.27) and (3.30) gives (3.23). □

4. Intergrid operators. In this section we construct grid transfer operators $I_H^h : W_H \to W_h$ and $J_h^H : W_h \to W_H$ for which properties A.1 and A.3 are subsequently proved. As is done by Brenner for Morley spaces in [4], we define I_H^h and J_h^H as compositions:

$$I_H^h = F_h \circ E_H \tag{4.1}$$

$$J_h^H = F_H \circ Q_h^H \circ E_h \tag{4.2}$$

where E_h (E_H) is a particular injection of the piecewise-polynomial space W_h (W_H) into the C^1-quintic Argyris space, to be denoted $\tilde{\mathcal{W}}_h$ $(\tilde{\mathcal{W}}_H)$, cf. [7], F_h (F_H) is the standard W_h nodal interpolation operator, and $Q_h^H : \mathcal{W}_h \to \mathcal{W}_H$ is the L_2-orthogonal projection associated with the somewhat larger spaces

$$\mathcal{W}_h = \left\{ w \in C^1(\overline{\Omega}) : w|_T \in \mathcal{P}_5(T) \qquad \forall\, T \in \mathcal{T}_h \quad \text{and} \quad w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\} \tag{4.3}$$

satisfying $\mathcal{W}_h \supseteq \tilde{\mathcal{W}}_h$, with $\mathcal{W}_H$ similarly defined.

More specifically, the operator $E_h : W_h \to \tilde{\mathcal{W}}_h$ (similarly E_H) is defined by:

$$\begin{array}{ll} (a) & (E_h w)(p_i) = w(p_i) \quad \text{at all internal vertices } p_i \in \mathcal{T}_h, \\ (b) & \text{for } |\alpha| = 2,\ (\partial^\alpha E_h w)(p_i) = 0 \quad \text{at all internal vertices } p_i \in \mathcal{T}_h, \\ (c) & \text{for } |\alpha| = 1,\ (\partial^\alpha E_h w)(p_i) = \text{average of values } (\partial^\alpha w)(p_i) \\ & \qquad\qquad \text{at all internal vertices } p_i \in \mathcal{T}_h, \\ (d) & (\frac{\partial}{\partial\nu} E_h w)(m_j) = \text{average of values } (\frac{\partial}{\partial\nu} w)(m_j) \\ & \qquad\qquad \text{at all internal midpoints } m_j \in \mathcal{T}_h, \text{ and} \\ (e) & \text{nodal values of } E_h w \text{ and its derivatives are zero on } \partial\Omega. \end{array} \tag{4.4}$$

This is essentially the same injection proposed for Morley spaces in [4], except for the average at each midpoint (4.4)(d) which is now needed since the normal derivative of $w \in W_h$ typically jumps at edge midpoints. Of course the first derivative values of both Morley and C^0 piecewise-polynomial elements typically jump at each internal vertex as well, hence the average in (4.4)(c).

With I_H^h and J_h^H defined in this way, properties A.1 and A.3 are established with the aid of the following three lemmas, cf. [4].

LEMMA 4.1. *For all $w \in \mathcal{W}_h$*

$$\left|Q_h^H w\right|_2 \le C|w|_2, \quad \text{and} \tag{4.5}$$

$$\left\|w - Q_h^H w\right\|_0 + H\left|w - Q_h^H w\right|_1 \le C H^2 |w|_2\,. \tag{4.6}$$

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LEMMA 4.2. *For all* $w \in \mathcal{W}_h$

$$\|F_h w\|_{2,h} \leq C|w|_2, \quad \text{and} \tag{4.7}$$

$$\|w - F_h w\|_0 + h|w - F_h w|_1 \leq Ch^2 |w|_2 \, . \tag{4.8}$$

LEMMA 4.3. *For all* $w \in W_h$

$$|E_h w|_2 \leq C\|w\|_{2,h}, \quad \text{and} \tag{4.9}$$

$$\|w - E_h w\|_0 + h|w - E_h w|_1 \leq Ch^2 \|w\|_{2,h} \, . \tag{4.10}$$

Similar estimates with each h replaced by H also hold for both F_H and E_H.

The proof of Lemma 4.1 is given by Brenner (Lemma 4.1, [4]). The estimate (4.8) for the nodal interpolant F_h is a standard result, cf. [7]. As with Assumption A.4a, (4.7) can be established using a trace inequality and scaling argument like that used to prove Lemma 4 in [2]. Here we need only establish the following

Proof. (of Lemma 4.3) For $w \in W_h$, we begin by proving a local estimate on $T \in \mathcal{T}_h$. We denote $w_T = w|_T$ and $\overline{w}_T = E_h w|_T$, and refer to the vertices p_i, the midpoints m_j, and the diameter h_T of triangle T.

The difference $(w - E_h w)|_T = w_T - \overline{w}_T \in \mathcal{P}_5(T)$ can be expressed as a sum of (a subset of) nodal basis functions for the Argyris finite element

$$w_T - \overline{w}_T = \sum_{i=1}^{3} \sum_{|\alpha|=1,2} \partial_\alpha (w_T - \overline{w}_T)(p_i)\, r_{\alpha,i} + \sum_{j=1}^{3} \frac{\partial}{\partial \nu}(w_T - \overline{w}_T)(m_j)\, q_j \tag{4.11}$$

where the basis functions $r_{\alpha,i}$ and q_j corresponding to the nodal variables $(\partial_\alpha v)(p_i)$ and $\frac{\partial v}{\partial \nu}(m_j)$, respectively, satisfy

$$\|r_{\alpha,i}\|_{0,T} \leq Ch_T^{1+|\alpha|}, \quad \text{and} \tag{4.12}$$

$$\|q_j\|_{0,T} \leq Ch_T^2 \, . \tag{4.13}$$

The estimates (4.12) and (4.13) can be obtained by projecting the basis functions onto the affine-equivalent C^0-quintic Hermite triangle and using a scaling argument, see [7].

Noting that $\overline{w}_T$ and the jump operator $\mathcal{J}$ are defined so that $\frac{\partial}{\partial \nu}(w_T - \overline{w}_T)(m_j) = \frac{1}{2}\mathcal{J}\frac{\partial w}{\partial \nu}(m_j)$, even when m_j is on $\partial\Omega$, and by virtue of the simple quadrature result

$$\int_e f(s)\, ds = |e| f(m_j) + \frac{1}{24}|e|^3 f''(\alpha) \qquad \text{for some } \alpha \in e, \text{ if } f \in C^2(e), \tag{4.14}$$

we have

$$\begin{aligned}
\left| \frac{\partial}{\partial \nu}(w_T - \overline{w}_T)(m_j) \right| &= \frac{1}{2} \left| \mathcal{J}\frac{\partial w}{\partial \nu}(m_j) \right| \\
&\leq \frac{1}{2} \left(\frac{1}{|e|} \int_e \mathcal{J}\frac{\partial w}{\partial \nu}\, ds + \frac{1}{24}\,|e|^2 \left\| \mathcal{J}\tfrac{\partial w}{\partial \nu} \right\|_{2,\infty,e} \right) \\
&\leq \frac{1}{2} \left(\int_e (|e|^{-1/2})^2\, ds \right)^{1/2} \left(\int_e (|e|^{-1/2}\mathcal{J}\frac{\partial w}{\partial \nu})^2\, ds \right)^{1/2} \\
&\quad + \frac{1}{48}\,|e|^2 \left\| \mathcal{J}\tfrac{\partial w}{\partial \nu} \right\|_{2,\infty,e}
\end{aligned} \tag{4.15}$$

$$\leq \frac{1}{2}\left(\frac{1}{|e|}\int_e \left|\mathcal{J}\frac{\partial w}{\partial \nu}\right|^2 ds\right)^{1/2} + \frac{1}{48}|e|^2 \left\|\mathcal{J}\tfrac{\partial w}{\partial \nu}\right\|_{2,\infty,e}$$
$$\leq C\left[\left(\frac{1}{h}\int_e \left|\mathcal{J}\frac{\partial w}{\partial \nu}\right|^2 ds\right)^{1/2} + h_T^2 \left\|\mathcal{J}\tfrac{\partial w}{\partial \nu}\right\|_{2,\infty,e}\right]$$

having made use of the Schwarz inequality, and $1/|e| \leq c/h_T \leq C/h$, which is a consequence of the minimum angle condition (regularity), and the quasiuniformity of $\mathcal{T}_h$.

In order to estimate $\partial_\alpha(w_T - \overline{w}_T)(p_i)$ for $|\alpha| = 1$, let $\mathcal{T}_{p_i}$ denote the set of triangles $T \in \mathcal{T}_h$ which contain vertex p_i. Then

$$\partial_\alpha(w_T - \overline{w}_T)(p_i) = \frac{1}{|\mathcal{T}_{p_i}|}\sum_{\tilde{T}\in\mathcal{T}_{p_i}} \partial_\alpha(w_T - w_{\tilde{T}})(p_i) \tag{4.16}$$

where $w_{\tilde{T}} = w|_{\tilde{T}}$. Although T and $\tilde{T}$ may not be adjacent triangles, $w_T - w_{\tilde{T}}$ (and ∂_α at p_i) can be expressed as a sum of differences involving adjacent triangles, say

$$w_T - w_{\tilde{T}} = (w_T - w_{T_1}) + (w_{T_1} - w_{T_2}) + \cdots + (w_{T_j} - w_{\tilde{T}}) \tag{4.17}$$

where the sum contains at most $\lfloor|\mathcal{T}_{p_i}|/2\rfloor$ terms. Consequently, it is sufficient to estimate $\partial_\alpha(w_{T_k} - w_{T_{k+1}})(p_i)$ where T_k and T_{k+1} share an edge e_k having p_i as one endpoint.

Since $w_{T_k} - w_{T_{k+1}} = 0$ along each edge e_k, the component of $\partial_\alpha(w_{T_k} - w_{T_{k+1}})$ tangential to e_k is zero. Considering next the normal component of this derivative (considered as a function of arclength s along e_k with midpoint m_k), we denote

$$N(s) = \frac{\partial}{\partial \nu}\left(w_{T_k} - w_{T_{k+1}}\right)(s)$$

and recall that $N(m_k) = \frac{1}{2}\mathcal{J}\frac{\partial w}{\partial \nu}(m_k)$. Since

$$\left\|\mathcal{J}\tfrac{\partial w}{\partial \nu}\right\|_{2,\infty,e} \leq \|w\|_{3,\infty,T_k} + \|w\|_{3,\infty,T_{k+1}} \tag{4.18}$$

(4.15), (4.18), and the Mean Value Theorem give

$$\begin{aligned}
&\left|\frac{\partial}{\partial \nu}\left(w_{T_k} - w_{T_{k+1}}\right)(p_i)\right| \\
&\quad = |N(p_i)| \\
&\quad \leq |N(m_k)| + \left|w_{T_k} - w_{T_{k+1}}\right|_{2,\infty,e_k}\left(\frac{1}{2}|e_k|\right) \\
&\quad \leq C\left[\left(h^{-1}\int_{e_k}\left|\mathcal{J}\frac{\partial w}{\partial \nu}\right|^2 ds\right)^{1/2} + h_{T_k}^2\left(\|w\|_{3,\infty,T_k} + \|w\|_{3,\infty,T_{k+1}}\right)\right. \\
&\qquad \left. + h_{T_k}\left(|w|_{2,\infty,T_k} + |w|_{2,\infty,T_{k+1}}\right)\right] .
\end{aligned} \tag{4.19}$$

By virtue of the quasiuniformity of $\mathcal{T}_h$, combining (4.15), (4.16), (4.18), (4.17), (4.19), and the standard inverse estimates

$$|w|_{2+l,\infty,T} \leq C(T)\, h_T^{-1-l}\, |w|_{2,T} \qquad l = 0, 1\,, \tag{4.20}$$

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yields

$$\sum_{|\alpha|=1} |\partial_\alpha (w_T - \overline{w}_T)(p_i)| \tag{4.21}$$
$$\leq C \left[\sum_{e \in \Gamma_{p_i}} \left(h^{-1} \int_e \left| \mathcal{J} \frac{\partial w}{\partial \nu} \right|^2 \, ds \right)^{1/2} + \sum_{T' \in \mathcal{T}_{p_i}} |w|_{2,T'} \right] ,$$

where Γ_{p_i} denotes the set of edges of $\mathcal{T}_h$ (and $\mathcal{T}_{p_i}$) incident upon vertex p_i. Combining (4.15), (4.18), and (4.20), we also have

$$\left| \frac{\partial}{\partial \nu} (w_T - \overline{w}_T)(m_j) \right| \leq C \left[\left(h^{-1} \int_e \left| \mathcal{J} \frac{\partial w}{\partial \nu} \right|^2 \, ds \right)^{1/2} + \sum_{T' \in \mathcal{T}_{p_i}} |w|_{2,T'} \right] , \tag{4.22}$$

where $\mathcal{T}_{m_j}$ denotes the pair of triangles which share the midpoint m_j.

For second derivatives ($|\alpha| = 2$) one has from (4.4b), (4.20), and the quasiuniformity of $\mathcal{T}_h$

$$|\partial_\alpha (w_T - \overline{w}_T)(p_i)| \; = \; |\partial_\alpha w_T(p_i)| \; \leq \; |w|_{2,\infty,T} \; \leq \; C \, h^{-1} \, |w|_{2,T} \, . \tag{4.23}$$

Combining (4.11)-(4.13), (4.22), (4.21), and (4.23), one obtains the local estimate

$$\|w_T - \overline{w}_T\|_{0,T}^2 \leq C \, h^4 \left[\frac{1}{h} \int_{\cup_i \Gamma_{p_i}} \left| \mathcal{J} \frac{\partial w}{\partial \nu} \right|^2 \, ds + \sum_{T' \in \cup_i \mathcal{T}_{p_i}} |w|_{2,T'}{}^2 \right] ,$$

from which follows

$$\|w - E_h w\|_0 \leq C \, h^2 \|w\|_{2,h} \, . \tag{4.24}$$

The remaining estimates in Lemma 4.3 follow from inverse properties and the triangle inequality. $\square$

With Lemmas 4.1-3 now established, property A.3a can be verified

$$\begin{aligned} \left\| J_h^H v \right\|_{2,H} &= \left\| F_H \{ Q_h^H [E_h v] \} \right\|_{2,H} \\ &\leq C \left| Q_h^H [E_h v] \right|_2 \\ &\leq C \, |E_h v|_2 \\ &\leq C \, \|v\|_{2,h} \, . \end{aligned} \tag{4.25}$$

Furthermore, Lemmas 4.1-3, the triangle inequality and (4.25) yield

$$\begin{aligned} \left\| J_h^H v - v \right\|_0 &\leq \left\| F_h [Q_h^H E_h v] - [Q_h^H E_h v] \right\|_0 + \left\| Q_h^H [E_h v] - [E_h v] \right\|_0 + \|E_h v - v\|_0 \\ &\leq C \, H^2 \left| Q_h^H E_h v \right|_2 + C \, H^2 |E_h v|_2 + C h^2 \|v\|_{2,h} \\ &\leq C \, H^2 \|v\|_{2,h} \, . \end{aligned}$$

The difference $\left| J_h^H v - v \right|_1$ may be similarly estimated and property A.3b follows. The proof of A.1 is similar.

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