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MODIFIED SPECHT'S PLATE BENDING ELEMENT AND ITS CONVERGENCE ANALYSIS *

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Abstract. This paper discusses Specht's plate bending element, shows the relationships between $\int_{F_\rho} w\,ds$ or $\int_{F_\rho} \frac{\partial w}{\partial n}\,ds$ and the nodal parameters (or degrees of freedom), further it sheds lights on the construction methods for that element, and finally it introduces a new plate bending element with good convergent properties (which passes the F-E-M-Test (cf.[11])) is derived.

Key words. interpolation, nonconforming finite element, Specht's element.

AMS subject classifications. 41A05, 65D05, 65N30.

1. Introduction.

The solution with a C^1-continuity requirement of Kirchhoff bending using a finite element models results in complicated higher elements (cf.[2], [4], [7]). Besides the large number of unknowns, difficulties may also arise from mixed second derivatives at the vertices taken as nodal variables (cf.[8]). To overcome such difficulties, a splitting spline element method was introduced (cf.[5],[9]), but this result in a complicated computation. From the practical point of view lower-degree polynomial finite elements are more desirable. Unfortunately, the simple elements based on lower degree polynomials for the displacement field are non-conforming (not C^1 compatible). This may cause convergence problems and unreliable finite approximations. For non-conforming finite elements, one has some relaxed sufficient convergence conditions, such as the well-known patch test, the interpolation test, the generalized patch tests and the F-E-M-Test, instead of the strong C^1 continuity.

Consider the simple triangular plate bending element whose nodal variables (or degrees or freedom) are the deflection and two rotations at the vertices. Based on a quadratic displacement expansion proposed by Zienkiewicz, this element is nonconforming because the normal slopes do not match continously along the interelement boundaries. As this element fails in the (generalized) patch test (cf.[10]), Bergan in [1] proposed a modified displacement basis, but the modified version does not satisfy the patch test either. Later, with the aid of the interpolation test, B. Specht (cf.[13]) constructed an appropriate polynomial displacement basis. This modified element passes the (generalized) patch test ensuring the convergence.

Specht's construction is based on the requirement of weak continuity, i.e., the displacement w and the normal slope $\frac{\partial w}{\partial n}$ (and tangent slope $\frac{\partial w}{\partial \tau}$) are continuous in the integral sense along the interelement boundaries. The intention of this article is to derive the relationships between $\int_{F_\rho} w\,ds$ as well as $\int_{F_\rho} \frac{\partial w}{\partial n}\,ds$ and the nodal variables, to examine a constructive method for Specht's plate bending element, and to introduce a new plate bending element with convergence by the aid of Shi's F-E-M-Test (cf.[11]).

To facilitate our presentation, we must agree on certain notations. Given a triangle K with the vertices $P_i = (x_i, y_i)(i = 1, 2, 3)$ in counterclockwise order and the

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area Δ, we put

$$\begin{array}{c} \xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2 \\ \eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1, \quad \eta_3 = y_1 - y_2 \\ l_{12}^2 = \xi_3^2 + \eta_3^2, \quad l_{23}^2 = \xi_1^2 + \eta_1^2, \quad l_{31}^2 = \xi_2^2 + \eta_2^2 \\ r_1 = \frac{1}{\Delta}(\xi_2\xi_3 + \eta_2\eta_3), \; r_2 = \frac{1}{\Delta}(\xi_3\xi_1 + \eta_3\eta_1), \; r_3 = \frac{1}{\Delta}(\xi_1\xi_2 + \eta_1\eta_2) \\ t_1 = \frac{1}{\Delta}(\xi_1^2 + \eta_1^2), \quad t_2 = \frac{1}{\Delta}(\xi_2^2 + \eta_2^2), \quad t_3 = \frac{1}{\Delta}(\xi_3^2 + \eta_3^2) \end{array}.$$

Denote by F_i the edge of K opposite to the vertex P_i, and by τ_i and n_i the unit tangent and outward normal on $F_i (i = 1, 2, 3)$, respectively. Now we let λ_i denote the area coordinates relative to the vertices P_i, i.e.,

$$\begin{cases} x = x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 \\ y = y_1\lambda_1 + y_2\lambda_2 + y_3\lambda_3 \\ 1 = \lambda_1 + \lambda_2 + \lambda_3 \end{cases}$$

such that the triangle K is transformed into the standard simplex $K^* = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 + \lambda_2 + \lambda_3 = 1, \;\; \lambda_i \geq 0\}$.

2. Analysis for Specht's element. Specht's plate bending element was defined in [13] as follows. Let K be a triangle with vertices at $P_i = (x_i, y_i), (i = 1, 2, 3)$ in counterclockwise order. Specht's element has three degrees of freedom per vertex, i.e., displacement at vertex and the two rotations expressed by the derivatives of the transverse displacement, similar to Zienkiewicz's element,

$$\begin{array}{rl} D(K,w) = & (w(P_1), w_x(P_1), w_y(P_1), w(P_2), w_x(P_2), w_y(P_2), \\ & w(P_3), w_x(P_3), w_y(P_3))^T \end{array} \tag{2.1}$$

The shape function space of Specht's element is

$$P(K) = \{w \in R(K) | \int_{F_i} P_2^{(i)} \frac{\partial w}{\partial n_i}\, ds = 0, \quad 1 \leq i \leq 3\}, \tag{2.2}$$

where $P_2^{(i)}$ is the Legendre polynomial of order 2 on F_i and

$$\begin{array}{rl} R(K) = & \text{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \\ & \lambda_3^2\lambda_1, \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2\}. \end{array} \tag{2.3}$$

It is clear that dim$P(K) = 9$ and the interpolation problem $(P(K), D(K,w))$ is unisolvable, i.e., for any given constants $C = (c_1, c_2, \cdots, c_9)^T$ there exists unique $w \in P(K)$ such that $D(K,w) = C$. In [13], B. Specht wrote "The required three higher terms are assumed linear combinations of the following cubic and quartic terms: $\lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2$. This assumption is successful", but why did B.Specht add those terms? To explain Specht's element again, first, we introduce an interpolation theorem.

Let $\pi_k(K)$ be polynomial space of order k defined on K, and denote by $\Lambda(K,w)$ the following interpolation conditions (or linear functionals defined on $\pi(K)$)

$$\begin{array}{rl} \Lambda(K,w) = & (w(P_1), w_x(P_1), w_y(P_1), w(P_2), w_x(P_2), w_y(P_2), \\ & w(P_3), w_x(P_3), w_y(P_3), \int_{F_1} w\, ds, \int_{F_2} w\, ds, \\ & \int_{F_3} w\, ds, \int_{F_1} \frac{\partial w}{\partial n_1}\, ds, \int_{F_2} \frac{\partial w}{\partial n_2}\, ds, \int_{F_3} \frac{\partial w}{\partial n_3}\, ds)^T \end{array}. \tag{2.4}$$

THEOREM 2.1. *The interpolation problem* $(\pi_4(K), \Lambda(K, w))$ *is unisolvable, that is, for any given constants* $C = (c_1, c_2, \cdots, c_{15})^T$*, there exists a unique polynomial* $w \in \pi_4(K)$ *such that*

$$\Lambda(K, w) = C.$$

Proof. For $w \in \pi_4(K)$, by the Bernstein-Bezier representation, we have

$$w = \sum_{i+j+k=4} \frac{4!}{i!j!k!}\, w_{ijk}\, \lambda_1^i \lambda_2^j \lambda_3^k.$$

It is not difficult to show that the coefficients w_{ijk} $(i = 0$ or $j = 0$ or $k = 0)$ can be represented by $w(P_1) = c_1, w_x(P_1) = c_2, w_y(P_1) = c_3,\ w(P_2) = c_4, w_x(P_2) = c_5, w_y(P_2) = c_6,\ w(P_3) = c_7, w_x(P_3) = c_8, w_y(P_3) = c_9,\ \int_{F_1} w\, ds = c_{10}, \int_{F_2} w\, ds = c_{11}, \int_{F_3} w\, ds = c_{12}$. □

By the aid of the barycentric coordinates with respect to K, we obtain

$$\begin{aligned} l_{12}\frac{\partial w}{\partial n_3} &= -\tfrac{1}{2}\left(r_2\frac{\partial w}{\partial \lambda_1} + r_1\frac{\partial w}{\partial \lambda_2} + t_3\frac{\partial w}{\partial \lambda_3}\right) \\ &= -2\Bigg(r_2 \sum_{i+j+k=3} \tfrac{3!}{i!j!k!}\, w_{i+1\,jk}\lambda_1^i\lambda_2^j\lambda_3^k \\ &\qquad + r_1 \sum_{i+j+k=3} \tfrac{3!}{i!j!k!}\, w_{ij+1\,k}\lambda_1^i\lambda_2^j\lambda_3^k \\ &\qquad + t_3 \sum_{i+j+k=3} \tfrac{3!}{i!j!k!}\, w_{ijk+1}\lambda_1^i\lambda_2^j\lambda_3^k\Bigg). \end{aligned}$$

Substituting $\lambda_3 = 0$ on the edge P_1P_2 yields the following relation:

$$\begin{aligned} l_{12}\ \frac{\partial w}{\partial n_3}\bigg|_{\lambda_3=0} = \ & -2\Bigg(r_2 \sum_{i+j=3} \tfrac{3!}{i!j!}\, w_{i+1\,j0}\lambda_1^i\lambda_2^j \\ & + r_1 \sum_{i+j=3} \tfrac{3!}{i!j!}\, w_{ij+1\,0}\lambda_1^i\lambda_2^j \\ & + t_3 \sum_{i+j=3} \tfrac{3!}{i!j!}\, w_{ij1}\lambda_1^i\lambda_2^j\Bigg). \end{aligned}$$

Integrating the above equation on the edge F_3 yields

$$\begin{aligned} & 2l_{12}\int_{F_3} \tfrac{\partial w}{\partial n_3}\, ds \\ = \ & -\left(r_2 \sum_{i+j=3} w_{i+1j0} + \sum_{i+j=3} w_{ij+10} + t_3 \sum_{i+j=3} w_{ij1}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} t_3(w_{211} + w_{121}) = \ & -2l_{12}\int_{F_3} \tfrac{\partial w}{\partial n_3}\, ds - t_3(w_{301} + w_{031}) \\ & - r_2 \sum_{i+j=3} w_{i+1j0} - r_1 \sum_{i+j=3} w_{ij+10} \quad . \end{aligned}$$

Similarily, the following relations are derived:

$$\begin{aligned} t_1(w_{121} + w_{112}) = \ & -2l_{23}\int_{F_1} \tfrac{\partial w}{\partial n_1}\, ds - t_1(w_{130} + w_{103}) \\ & - r_3 \sum_{j+k=3} w_{0j+1k} - r_2 \sum_{j+k=3} w_{0jk+1} \end{aligned}$$

$$
\begin{aligned}
t_2(w_{211}+w_{112}) = \quad & -2l_{31}\int_{F_2}\tfrac{\partial w}{\partial n_2}\,ds - t_2(w_{301}+w_{103}) \\
& -r_1\sum_{i+k=3} w_{i0k+1} - r_3\sum_{i+k=3} w_{i+10k}\quad .
\end{aligned}
$$

Hence the coefficients w_{211}, w_{121} and w_{112} can be represented by $C = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15})^T$.

Now we consider the following interpolation problem: Find a 9-dimensional subspace $Q(K)$ of $\pi_4(K)$ such that for any given values $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ there exists a unique $w \in Q(K)$ satisfying

$$
\left\{
\begin{array}{lll}
w(P_1) = c_1, & w_x(P_1) = c_2, & w_y(P_1) = c_3, \\
w(P_2) = c_4, & w_x(P_2) = c_5, & w_y(P_2) = c_6, \\
w(P_2) = c_7, & w_x(P_3) = c_8, & w_y(P_3) = c_9,
\end{array}
\right. \tag{2.5}
$$

$$
\left\{
\begin{aligned}
\int_{F_3} w\,ds &= \tfrac{l_{12}}{2}[w(P_1)+w(P_2)] + \tfrac{l_{12}^2}{12}\left[\tfrac{\partial w}{\partial \tau_3}(P_1) - \tfrac{\partial w}{\partial \tau_3}(P_2)\right] \\
&= \tfrac{l_{12}}{2}[c_1+c_2] + \tfrac{l_{12}}{12}\left[\xi_3(c_5-c_2)+\eta_3(c_6-c_3)\right], \\
\int_{F_1} w\,ds &= \tfrac{l_{23}}{2}[w(P_2)+w(P_3)] + \tfrac{l_{23}^2}{12}\left[\tfrac{\partial w}{\partial \tau_1}(P_2) - \tfrac{\partial w}{\partial \tau_1}(P_3)\right] \\
&= \tfrac{l_{23}}{2}[c_2+c_3] + \tfrac{l_{23}}{12}\left[\xi_1(c_8-c_5)+\eta_1(c_9-c_6)\right], \\
\int_{F_2} w\,ds &= \tfrac{l_{31}}{2}[w(P_3)+w(P_1)] + \tfrac{l_{31}^2}{12}\left[\tfrac{\partial w}{\partial \tau_2}(P_3) - \tfrac{\partial w}{\partial \tau_2}(P_1)\right] \\
&= \tfrac{l_{31}}{2}[c_3+c_1] + \tfrac{l_{31}}{12}\left[\xi_2(c_2-c_8)+\eta_2(c_3-c_9)\right],
\end{aligned}
\right. \tag{2.6}
$$

and

$$
\left\{
\begin{aligned}
\int_{F_3} \tfrac{\partial w}{\partial n_3}\,ds &= \tfrac{l_{12}}{2}\left[\tfrac{\partial w}{\partial n_3}(P_1) + \tfrac{\partial w}{\partial n_3}(P_2)\right] \\
&= \tfrac{1}{2}[-\xi_3(c_2+c_5)+\eta_3(c_3+c_6)], \\
\int_{F_1} \tfrac{\partial w}{\partial n_1}\,ds &= \tfrac{l_{23}}{2}\left[\tfrac{\partial w}{\partial n_1}(P_2) + \tfrac{\partial w}{\partial n_1}(P_3)\right] \\
&= \tfrac{1}{2}[-\xi_1(c_5+c_8)+\eta_1(c_6+c_9)], \\
\int_{F_2} \tfrac{\partial w}{\partial n_2}\,ds &= \tfrac{l_{31}}{2}\left[\tfrac{\partial w}{\partial n_2}(P_3) + \tfrac{\partial w}{\partial n_2}(P_1)\right] \\
&= \tfrac{1}{2}[-\xi_2(c_8+c_2)+\eta_2(c_9+c_3)].
\end{aligned}
\right. \tag{2.7}
$$

Denoting by Q_1 the coefficient matrix, with respect to $C = (c_1, c_2, \cdots, c_9)^T$, of the right hand sides of (2.6) and (2.7), and letting $Q = \begin{pmatrix} I \\ Q_1 \end{pmatrix}$, then (2.5), (2.6) and (2.7) can be written as

$$
\Lambda(K, w) = Q\,D(K, w). \tag{2.8}
$$

Let the interpolation polynomial be

$$
w = \sum_{i+j+k=4} \frac{4!}{i!j!k!}\, w_{ijk}\, \lambda_1^i \lambda_2^j \lambda_3^k. \tag{2.9}
$$

Substituting (2.9) into (2.4) yields the following relationship

$$
\Lambda(K, w) = G\,X, \tag{2.10}
$$

where $X = (w_{ijk})^T_{i+j+k=4}$. Clearly G is a nonsingular matrix of order 15, in view of Theorem 2.1. Then according to (2.8) and (2.10), we have

$$
G\,X = Q\,D(K, w),
$$

Defining

$$Q(K)=\left\{w=\sum_{i+j+k=4}\frac{4!}{i!j!k!}\,w_{ijk}\,\lambda_1^i\lambda_2^j\lambda_3^k\in\pi_4(K)|\;GX=QD(K,w)\right\},$$

we obtain the following results.

THEOREM 2.2. *With assumptions as above we have $Q(K)=P(K)$ (The shape function space of Specht's element), and $(Q(K),D(K,w),K)$ is just the Specht's plate bending element.*

Proof. It is necessary to show that for any polynomial $w\in P(K)$, (2.6) and (2.7) are valid. In [12], Shi and Chen have showed that the integrals of normal slopes of Specht's element on each edge of K are discretized by a linear integral formula. Thus (2.7) is valid for Specht's element. Let $w\in P(K)$, then from [13] w is a polynomial of order 3 on each edge of K. Hence equations (2.6) are also valid for w. This completes the proof. ▯

By (2.6) and (2.7), element $(Q(K),D(K,w),K)$ ($K\in\Delta$ a triangulation) passes the strong F1 and strong F2 test (cf.[11]) which ensures convergence.

3. A new plate bending triangular element. It is known that the strong F1 and the strong F2 tests ensure the Patch Test for the plate bending problem, but the strong F1 and the strong F2 tests are indeed stronger conditions for the convergence of finite element. In general, the F1 test (not the strong F1 test) can be satisfied when the displacement values at the vertices of the triangular element are used as the degrees of freedom (or parameters) of the finite element (cf.[11]). Thus it is not essential how to discretize integrals $\int_{F_\rho} w\,ds$ (such as (2.6) in the construction of Specht's element). It is important to keep the strong F2 test.

Now let us discuss another interpolation problem given as follows: Find a polynomial subspace $R(K)$ such that for any given constants $C=(c_1,c_2,\cdots,c_{12})^T$ there exists a unique polynomial $w\in R(K)$ satisfying the following interpolation conditions:

$$\left\{\begin{array}{lll} w(P_1)=c_1, & w_x(P_1)=c_2, & w_y(P_1)=c_3\\ w(P_2)=c_4, & w_x(P_2)=c_5, & w_y(P_2)=c_6\\ w(P_3)=c_7, & w_x(P_3)=c_8, & w_y(P_3)=c_9\\ \multicolumn{3}{l}{\int_{F_1}\frac{\partial w}{\partial n_1}ds=c_{10},\int_{F_2}\frac{\partial w}{\partial n_2}ds=c_{11},\int_{F_3}\frac{\partial w}{\partial n_3}ds=c_{13}.}\end{array}\right. \tag{3.1}$$

Let

$$\begin{array}{rl} F(K,w)= & (w(P_1),w_x(P_1),w_y(P_1),w(P_2),w_x(P_2),w_y(P_2),w(P_3),\\ & w_x(P_3),w_y(P_3),\int_{F_1}\frac{\partial w}{\partial n_1}ds,\int_{F_2}\frac{\partial w}{\partial n_2}ds,\int_{F_3}\frac{\partial w}{\partial n_3}ds)^T.\end{array} \tag{3.2}$$

We will use the method introduced in [6] to find the interpolation subspace $R(K)$. For notation, set

$$\begin{array}{rl} R(K)=\pi_3(K)\oplus & \{d_1(r_3\lambda_2+r_2\lambda_3+t_1\lambda_1)(\lambda_2^3+\lambda_3^3+3\lambda_1\lambda_2\lambda_3)\\ & +d_2(r_1\lambda_3+r_3\lambda_1+t_2\lambda_2)(\lambda_3^3+\lambda_1^3+3\lambda_1\lambda_2\lambda_3)\\ & +d_3(r_2\lambda_1+r_1\lambda_2+t_3\lambda_3)(\lambda_1^3+\lambda_2^3+3\lambda_1\lambda_2\lambda_3):\\ & t_1d_1+t_2d_2+t_3d_3=0,\ d_i\in R\}.\end{array} \tag{3.3}$$

Referring to [6], in Section 5, we prove the following:

THEOREM 3.1. *The interpolation problem $(R(K),F(K,w),K)$ is unisolvable and* $\dim R(K)=12$.

Now let the interpolation polynomial be

$$
\begin{aligned}
w = \;& \beta_1\lambda_1^3 + \beta_2\lambda_2^3 + \beta_3\lambda_3^3 + \beta_4\lambda_1^2\lambda_2 + \beta_5\lambda_2^2\lambda_1 \\
& \beta_6\lambda_2^2\lambda_3 + \beta_7\lambda_3^2\lambda_2 + \beta_8\lambda_3^2\lambda_1 + \beta_9\lambda_1^2\lambda_3 + \beta_{10}\lambda_1\lambda_2\lambda_3 \\
& + d_1(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(\lambda_2^3 + \lambda_3^3 + 3\lambda_1\lambda_2\lambda_3) \\
& + d_2(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(\lambda_3^3 + \lambda_1^3 + 3\lambda_1\lambda_2\lambda_3) \\
& + d_3(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(\lambda_1^3 + \lambda_2^3 + 3\lambda_1\lambda_2\lambda_3)
\end{aligned} \tag{3.4}
$$

and $t_1d_1 + t_2d_2 + t_3d_3 = 0$. Substituting (3.4) into (3.2), we have

$$F(K, w) = C_{12\times 13}X$$

where $X = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, d_1, d_2, d_3)^T$ and $t_1d_1 + t_2d_2 + t_3d_3 = 0$, or $\begin{pmatrix} F(K,w) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix} X$ where $t = (0, 0, 0, 0, 0, 0, 0, 0, 0, t_1, t_2, t_3)$, then $\begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix}$ is a nonsingular matrix from Theorem 3.1. Now if we discretize the three integrals in (3.1) as in (2.7), then we have

$$F(K, w) = G\,D(K, w)$$

or

$$\begin{pmatrix} G\,D(K,w) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix} X,$$

and hence

$$X = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix}^{-1} \begin{pmatrix} G\,D(K,w) \\ 0 \end{pmatrix}. \tag{3.5}$$

Let $P^*(K) = \{w \in R(K) \mid w$ is defined as (3.4) and (3.5) and $D(K, w)$ is the degree of freedom $\}$. Then we have

THEOREM 3.2. *The new finite element $(P^*(K), D(K, w), K)$ passes the F1 test and the strong F2 test, and hence it converges for the plate bending problem. $P^*(K) \neq P(K)$ (shape function space of Specht's element).*

Proof. For $(P(K), D(K, w), K)$, integral $\int_{F_i} \frac{\partial w}{\partial n_i}\,ds$ depends only upon the parameters on the edge F_i in the sense of (2.7). Thus $\int_{F_i} \Delta \frac{\partial w}{\partial n_i}\,ds = 0$ along the interelement boundary F_i. On the other hand, as the values at the vertices of the triangles are degree of freedoms, we can easily prove that $\int_{F_i} \Delta\, w\, ds = o(||h||_{2,K_1\cup K_2})$, where F_i is the common boundary of K_1 and K_2. This is just the F1 test. With the conclusions of [11] the finite element $(P^*(K), D(K, w), K)$ is convergent over any regular triangulation for the fourth order elliptic problems. Finally, it is not difficult to show $P^*(K) \neq P(K)$ by direct computation. ☐

4. Analysis for Convergent Orders. Consider the clamped plate problem, which corresponds to the following data: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = f(v), \qquad \forall v \in H_0^2(\Omega), \tag{4.1}$$

with

$$a(u, v) = \int_\Omega \{\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\}\,dxdy,$$

and

$$f(v) = \int_{\Omega} f \cdot v \, dxdy,$$

where the constant σ (the Poisson coefficient of the material of which the plate is composed) lies in the interval $(0, \frac{1}{2})$.

For simplicity, we shall assume that the domain Ω is polygonal, so that it may be covered by a triangulation Δ which satisfies the ordinary regular conditions. Let h_K be the diameter of the triangular finite K and $h = \max_K h_K$.

Now construct the modified Specht's element introduced in the proceding section on each triangle K of the triangulation. Consequently a finite element space X_h on Ω can be obtained by standard method. Let

$$V_h = \{\ v_h \in X_h,\ {v_h}'\text{s parameters on } \partial\Omega \text{ are zero}\}.$$

Then the variational problem (4.1) can be discretized as: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = f(v_h), \qquad \forall v_h \in V_h, \tag{4.2}$$

where

$$a_h(u, v) = \sum_K \int_K \{\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\} \, dxdy.$$

Denote

$$|v_h|_{2,h} = (\sum_K |v_h|_{2,K}^2)^{\frac{1}{2}}.$$

First, we prove that $|v_h|_{2,h}$ is a norm of the finite element space V_h, i.e., $v_h \in V_h$ and $|v_h|_{2,h} = 0$ imply $v_h \equiv 0$. In fact, if $|v_h|_{2,h} = 0$, then $|v_h|_{2,K} = 0$ for each triangular element K. Hence $\frac{\partial v_h}{\partial x}$ and $\frac{\partial v_h}{\partial y}$ are constants, respectively, on each triangle. Let K_0 be a boundary triangle satisfying $F = K_0 \cap \partial\Omega$. Since the parameters of v_h are zero on F, one has

$$\int_F \frac{\partial v_h}{\partial s} \, ds \;=\; \int_F \frac{\partial v_h}{\partial n} \, ds \;=\; 0,$$

that is,

$$\int_F \frac{\partial v_h}{\partial x} \, ds \;=\; \int_F \frac{\partial v_h}{\partial y} \, ds \;=\; 0.$$

Hence

$$\frac{\partial v_h}{\partial x} = \frac{\partial v_h}{\partial y} = 0 \quad \text{on} \quad K_0,$$

as $\frac{\partial v_h}{\partial x}$ and $\frac{\partial v_h}{\partial y}$ are constants on K. On the other hand, the modified Specht's element passes the strong F2 test; hence we can conclude that $\frac{\partial v_h}{\partial x} = \frac{\partial v_h}{\partial y} = 0$ on each triangle K. Consequently, v_h is constant on K. Finally, the value of v_h at each boundary vertex is zero; hence $v_h \equiv 0$.

Assume that u and u_h are the solutions of the variational problems (4.1) and (4.2), respectively. To estimate the error between the finite element solution u_h and exact solution u is the next goal of ours.

THEOREM 4.1. *Let* $u \in H^4(\Omega) \cap H^2_0(\Omega)$. *Then*

$$|u - u_h|_{2,h} \le Ch(|u|_3 + h|u|_4), \tag{4.3}$$

$$|u - u_h|_{0,h} \le Ch^2(|u|_3 + h|u|_4). \tag{4.4}$$

Proof. From the well-known Strang's Lemma, one has

$$|u - u_h|_{2,h} \le C\left(\inf_{v_h \in V_h} |u - v_h|_{2,h} + \sup_{w_h \in V_h} \frac{|E_h(u, w_h)|}{|w_h|_{2,h}}\right), \tag{4.5}$$

where

$$E_h(u,w) = E_1(u,w) + E_2(u,w) + E_3(u,w),$$

$$E_1(u,w) = \sum_K \int_{\partial K} (\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2})\frac{\partial w}{\partial n}\, ds,$$

$$E_2(u,w) = \sum_K \int_{\partial K} (1-\sigma)\frac{\partial^2 u}{\partial n \partial s}\frac{\partial w}{\partial s}\, ds,$$

$$E_3(u,w) = -\sum_K \int_{\partial K} \frac{\partial(\Delta u)}{\partial n} w\, ds.$$

The first term on the right hand side of (4.5) is the approximation error. The second one is the consistency error.

In the following the two error terms are estimated respectively.

(I) The approximation error: $\inf_{v_h \in V_h} |u - v_h|_{2,h}$.

It is not difficult to prove that $\pi_2 \subset V_h$. This is because, for any quadratic polynomial u, (2.7) is exactly valid.

Define the interpolation operator $Q_h : v \in C^1(\overline{\Omega}) \to Q_h v \in V_h$ such that, on each triangle K,

$$Q_h v(P_i) = v(P_i), \quad (Q_h v)_x(P_i) = v_x(P_i), \quad (Q_h v)_y(P_i) = v_y(P_i), \quad i = 1,2,3,$$

$$\int_{F_3} \frac{\partial P_h v}{\partial n_3}\, ds = \frac{1}{2}[-\xi_3(v_x(P_1) + v_x(P_2)) + \eta_3(v_y(P_1) + v_y(P_2))],$$

$$\int_{F_1} \frac{\partial P_h v}{\partial n_1}\, ds = \frac{1}{2}[-\xi_1(v_x(P_2) + v_x(P_3)) + \eta_1(v_y(P_2) + v_y(P_3))],$$

and

$$\int_{F_2} \frac{\partial P_h v}{\partial n_2}\, ds = \frac{1}{2}[-\xi_2(v_x(P_3) + v_x(P_1)) + \eta_2(v_y(P_3) + v_y(P_1))].$$

According to the theory of Hermite interpolation, we have

$$\inf_{v_h \in V_h} |u - v_h|_{2,h} \le |u - Q_h u|_{2,h} \le Ch|u|_3.$$

(II) The consistency error: $\sup\limits_{w_h \in V_h} \frac{|E_h(u,w_h)|}{|w_h|_{2,h}}$.

By the construction of the modified Specht's element, $\int_F \frac{\partial w_h}{\partial s} ds$ and $\int_F \frac{\partial w_h}{\partial n} ds$ are continuous on any interelement boundary $F = K_1 \cap K_2$, and $\int_F \frac{\partial w_h}{\partial s} ds = \int_F \frac{\partial w_h}{\partial n} ds = 0$ on $F = F_0 \cap \partial\Omega$. On the other hand, we may construct a cubic Hermite interpolation $I_F w_h$ for w_h on each F (interelement edge and boundary edge) with finite element parameters. So $\int_F w_h\, ds$ can be computed by $\int_F I_F w_h\, ds$ with error term in higher orders of derivatives. Then applying the standard analytical techniques for nonconforming finite elements, the following estimations can be easily proved:

$$|E_i(u, w_h)| \le Ch|u|_3 \cdot |w_h|_{2,h}, \quad i = 1, 2,$$

$$|E_3(u, w_H)| \le Ch(|u|_3 + h|u|_4)|w_h|_{2,h}.$$

Hence

$$\sup_{w_h \in V_h} \frac{E_h(u, w_h)|}{|w_h|_{2,h}} \le Ch(|u|_3 + h|u|_4).$$

Thus (4.3) is obtained.

Finally, (4.4) can be proved by the Nitsche's technique or by the dual principle.
□

5. The Proof of Theorem 3.1.

In this section we will use the construction introduced in [6] to find the interpolation subspace $R(K)$ related to the interpolation conditions (3.1) or $F(K, w)$ of (3.2).

Let $w \in \pi(K)$ and $w(x, y) \equiv w(\lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3)$ is the berycentric coordinate of (x, y) with respect to the triangle K. We associate a function analytic at 0 with each interpolation condition of (3.1).

$$w(P_1) = w(1,0,0) \leftrightarrow e^{\lambda_1}, w(P_2) = w(0,1,0) \leftrightarrow e^{\lambda_2}, w(P_3) = w(0,0,1) \leftrightarrow e^{\lambda_3},$$

$$
\begin{array}{rcl}
w_x(P_1) & = & \frac{1}{2\Delta}\left[\eta_1\frac{\partial}{\partial\lambda_1}+\eta_2\frac{\partial}{\partial\lambda_2}+\eta_3\frac{\partial}{\partial\lambda_3}\right] w(1,0,0) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\eta_1\lambda_1+\eta_2\lambda_2+\eta_3\lambda_3)e^{\lambda_1} = l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}, \\
w_y(P_1) & = & -\frac{1}{2\Delta}\left[\xi_1\frac{\partial}{\partial\lambda_1}+\xi_2\frac{\partial}{\partial\lambda_2}+\xi_3\frac{\partial}{\partial\lambda_3}\right] w(1,0,0) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\xi_1\lambda_1+\xi_2\lambda_2+\xi_3\lambda_3)e^{\lambda_1} = l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}, \\
w_x(P_2) & = & \frac{1}{2\Delta}\left[\eta_1\frac{\partial}{\partial\lambda_1}+\eta_2\frac{\partial}{\partial\lambda_2}+\eta_3\frac{\partial}{\partial\lambda_3}\right] w(0,1,0) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\eta_1\lambda_1+\eta_2\lambda_2+\eta_3\lambda_3)e^{\lambda_2} = l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}, \\
w_y(P_2) & = & -\frac{1}{2\Delta}\left[\xi_1\frac{\partial}{\partial\lambda_1}+\xi_2\frac{\partial}{\partial\lambda_2}+\xi_3\frac{\partial}{\partial\lambda_3}\right] w(0,1,0) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\xi_1\lambda_1+\xi_2\lambda_2+\xi_3\lambda_3)e^{\lambda_2} = l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}, \\
w_x(P_3) & = & \frac{1}{2\Delta}\left[\eta_1\frac{\partial}{\partial\lambda_1}+\eta_2\frac{\partial}{\partial\lambda_2}+\eta_3\frac{\partial}{\partial\lambda_3}\right] w(0,0,1) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\eta_1\lambda_1+\eta_2\lambda_2+\eta_3\lambda_3)e^{\lambda_3} = l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}, \\
w_y(P_3) & = & -\frac{1}{2\Delta}\left[\xi_1\frac{\partial}{\partial\lambda_1}+\xi_2\frac{\partial}{\partial\lambda_2}+\xi_3\frac{\partial}{\partial\lambda_3}\right] w(0,0,1) \\
 & \leftrightarrow & \frac{1}{2\Delta}(\xi_1\lambda_1+\xi_2\lambda_2+\xi_3\lambda_3)e^{\lambda_3} = l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},
\end{array}
$$

$$
\begin{array}{rcl}
\int_{F_3}\frac{\partial w}{\partial n_3}\,ds & = & l_{12}\frac{\partial w}{\partial n_3}(P_1)+\frac{l_{12}^2}{2}\frac{\partial^2 w}{\partial\tau_3^2\partial n_3}(P_1)+\frac{l_{12}^3}{6}\frac{\partial^3 w}{\partial\tau_3^2\partial n_3}(P_1) \\
 & & +\frac{l_{12}^4}{24}\frac{\partial^4 w}{\partial\tau_3^3\partial n_3}(P_1)+\cdots \\
 & = & \left(1+\frac{1}{2}(\frac{\partial}{\partial\lambda_2}-\frac{\partial}{\partial\lambda_1})+\frac{1}{6}(\frac{\partial}{\partial\lambda_2}-\frac{\partial}{\partial\lambda_1})^2+\frac{1}{24}(\frac{\partial}{\partial\lambda_2}-\frac{\partial}{\partial\lambda_1})^3+\cdots\right) \\
 & & \times\left(r_2\frac{\partial}{\partial\lambda_1}+r_1\frac{\partial}{\partial\lambda_2}+t_3\frac{\partial}{\partial\lambda_3}\right)w(1,0,0) \\
 & \leftrightarrow & \left[1+\frac{1}{2}(\lambda_2-\lambda_1)+\frac{1}{6}(\lambda_2-\lambda_1)^2+\frac{1}{24}(\lambda_2-\lambda_1)^3+\cdots\right]\times \\
 & & (r_2\lambda_1+r_1\lambda_2+t_3\lambda_3)e^{\lambda_1} \\
 & \equiv & p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}.
\end{array}
$$

Similarily,

$$
\begin{array}{rcl}
\int_{F_1}\frac{\partial w}{\partial n_1}\,ds & \leftrightarrow & \left[1+\frac{1}{2}(\lambda_3-\lambda_2)+\frac{1}{6}(\lambda_3-\lambda_2)^2+\frac{1}{24}(\lambda_3-\lambda_2)^3+\cdots\right]\times \\
 & & (r_3\lambda_2+r_2\lambda_3+t_1\lambda_1)e^{\lambda_2} \\
 & \equiv & p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2},
\end{array}
$$

$$
\begin{array}{rcl}
\int_{F_2}\frac{\partial w}{\partial n_2}\,ds & \leftrightarrow & \left[1+\frac{1}{2}(\lambda_1-\lambda_3)+\frac{1}{6}(\lambda_1-\lambda_3)^2+\frac{1}{24}(\lambda_1-\lambda_3)^3+\cdots\right]\times \\
 & & (r_1\lambda_3+r_3\lambda_1+t_2\lambda_2)e^{\lambda_3} \\
 & \equiv & p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},
\end{array}
$$

Define

$$
\begin{array}{rcl}
H & = & \text{span}\{e^{\lambda_1},e^{\lambda_2},e^{\lambda_3},l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1},l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1},l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}, \\
 & & l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2},l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}, \qquad\qquad , \\
 & & p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1},p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2},p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}\}.
\end{array}
$$

and let$H_{\downarrow} = \text{span}\{f_{\downarrow} \mid f\in H\}$ where $f_{\downarrow}$ is the leading term of the Taylor's series of f in H. Then from the conclusions of [6] $H_{\downarrow}$ is an interpolation polynomial space with respect to $F(K,w)$.

Let f be any function in H, i.e.

$$
\begin{array}{rl}
f = & c_1e^{\lambda_1}+c_2e^{\lambda_2}+c_3e^{\lambda_3}+c_4l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}+c_5l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2} \\
 & +c_6l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}+c_7l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}+c_8l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2} \\
 & +c_9l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}+c_{10}p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}+c_{11}p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2} \\
 & +c_{12}p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}.
\end{array}
$$

We expand f as a power series at $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$ and let the coefficients of all of the cubic terms be zero. Then we have a linear system of ten equations in terms of $c_1, \cdots, c_{12}$ which yields the following relations

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$$

$$\begin{array}{ll} c_4 = -\frac{1}{3\Delta}[\eta_3 c_{10} + \eta_2 c_{12}], & c_7 = -\frac{1}{3\Delta}[\xi_3 c_{10} + \xi_2 c_{12}], \\ c_5 = -\frac{1}{3\Delta}[\eta_3 c_{10} + \eta_1 c_{11}], & c_8 = -\frac{1}{3\Delta}[\xi_3 c_{10} + \xi_1 c_{11}], \\ c_6 = -\frac{1}{3\Delta}[\eta_1 c_{11} + \eta_2 c_{12}], & c_9 = -\frac{1}{3\Delta}[\xi_1 c_{11} + \xi_2 c_{12}] \end{array}$$

where

$$t_3 c_{10} + t_1 c_{11} + t_2 c_{12} = 0$$

Then we can prove that the coefficients of $1, \lambda_1, \lambda_2, \lambda_3,\ \lambda_1^2, \lambda_2^2, \lambda_3^2,\ \lambda_1\lambda_2, \lambda_2\lambda_3,$ $\lambda_3\lambda_1$ are also zero and that f is of the following form (here only the quartic terms are written):

$$\begin{array}{rl} f = & \frac{1}{72} c_{10}(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_1^3 - \lambda_2^3) \\ & +\frac{1}{72} c_{11}(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(3\lambda_2^2\lambda_3 + 3\lambda_2\lambda_3^2 - \lambda_2^3 - \lambda_3^3) \\ & +\frac{1}{72} c_{12}(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(3\lambda_3^2\lambda_1 + 3\lambda_3\lambda_1^2 - \lambda_3^3 - \lambda_1^3) + \cdots \end{array}$$

where $t_3 c_{10} + t_1 c_{11} + t_2 c_{12} = 0$. Hence we have, noting that $\lambda_1 + \lambda_2 + \lambda_3 \equiv 1$,

$$\begin{array}{rcl} H_\downarrow & = & \pi_3 \oplus \{c_{10}(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_1^3 - \lambda_2^3) \\ & & +c_{11}(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(3\lambda_2^2\lambda_3 + 3\lambda_2\lambda_3^2 - \lambda_2^3 - \lambda_3^3) \\ & & +c_{12}(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(3\lambda_3^2\lambda_1 + 3\lambda_3\lambda_1^2 - \lambda_3^3 - \lambda_1^3) : \\ & & \mid \quad t_3 c_{10} + t_1 c_{11} + t_2 c_{12} = 0\} \\ & \equiv & \pi \oplus \{d_1(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(\lambda_2^3 + \lambda_3^3 + 3\lambda_1\lambda_2\lambda_3) \\ & & +d_2(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(\lambda_3^3 + \lambda_1^3 + 3\lambda_1\lambda_2\lambda_3) \\ & & +d_3(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(\lambda_1^3 + \lambda_2^3 + 3\lambda_1\lambda_2\lambda_3) : \\ & & \mid \quad t_1 d_1 + t_2 d_2 + t_3 d_3 = 0 : d_i \in R\} = P^*(K) \end{array}$$

Thus the interpolation problem $(P^*(K), F(K, w), K)$ is unisovable.

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