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DISPLACEMENT PRECONDITIONER FOR TOEPLITZ LEAST SQUARES ITERATIONS *

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Abstract. We consider the solution of least squares problems $\min ||b - Ax||_2$ by the preconditioned conjugate gradient (PCG) method, for $m \times n$ complex Toeplitz matrices A of rank n. A circulant preconditioner C is derived using the T. Chan optimal preconditioner for $n \times n$ matrices using the displacement representation of A^*A. This allows the fast Fourier transform (FFT) to be used throughout the computations, for high numerical efficiency. Of course A^*A need never be formed explicitly. Displacement–based preconditioners have also been shown to be very effective in linear estimation and adaptive filtering. For Toeplitz matrices A that are generated by 2π-periodic continuous complex-valued functions without any zeros, we prove that the singular values of the preconditioned matrix AC^{-1} are clustered around 1, for sufficiently large n. We show that if the condition number of A is of $O(n^\alpha)$, $\alpha > 0$, then the least squares conjugate gradient method converges in at most $O(\alpha \log n + 1)$ steps. Since each iteration requires only $O(m \log n)$ operations using the FFT, it follows that the total complexity of the algorithm is then only $O(\alpha m \log^2 n + m \log n)$. Conditions for *superlinear convergence* are given and numerical examples are provided illustrating the effectiveness of our methods.

Key words. circulant preconditioner, conjugate gradient,displacement representation, fast Fourier transform (FFT), Toeplitz operator.

AMS subject classifications. 65F10, 65F15.

1. Introduction.

An $m \times n$ matrix A is called a Toeplitz matrix if its entries are constant along each diagonal, i.e.,

$$A = [a_{j,k}] = [a_{j-k}]_{0 \le j \le m-1,\ 0 \le k \le n-1} .$$

Least squares problems

$$\min_x ||b - Ax||_2, \tag{1.1}$$

in which A is an $m \times n$ Toeplitz matrix, $m \ge n$, occur in a variety of applications, especially in signal and image processing. Since these problems arise in many important areas where there is need for computing solutions in near "real time", considerable effort has been devoted to developing fast algorithms for the solution of (1.1). Most of this work has focused on direct methods, such as the fast QR factorization algorithms of Bojanczyk, Brent and de Hoog [4], Chun and Kailath [11], Cybenko [13] and Sweet [26]. The stability properties of these algorithms are not well understood, see Bunch [5] for nonsingular Toeplitz systems, and Luk and Qiao for direct orthogonal factorization for least squares problems [22]. Almost all fast orthogonal factorization methods involve the square of the condition number of the data matrix in their error analyses [22]. Advantages and disadvantages of direct versus iterative methods for symmetric positive definite systems are described in detail in Linzer [20, 21].

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Here we consider the use of iterative methods, such as preconditioned conjugate gradients (PCG) for the solution of (1.1). Although the classical PCG algorithm applies only to Hermitian positive definite systems of equations, extensions to non-Hermitian, indefinite and least squares problems exist, cf. Freund, Golub and Nachtigal [15]. In particular, one can apply the classical PCG method to the factored form of the normal equations

$$A^*(b - Ax) = 0,$$

as in the PCGLS algorithm, cf. Björck [3]. Here A^* denotes the conjugate transpose. For completeness, we list the PCGLS algorithm here.

Algorithm PCG for Least Squares. *Let $x^{(0)}$ be an initial approximation to $Tx = b$, and let C be a given preconditioner. This algorithm computes the least squares solution, x, to $Tx = b$.*

$r^{(0)} = b - Tx^{(0)}$
$p^{(0)} = s^{(0)} = C^{-*}T^*r^{(0)}$
$\gamma_0 = \|s^{(0)}\|_2^2$
for $k = 0, 1, 2, \ldots$
  $q^{(k)} = TC^{-1}p^{(k)}$
  $\alpha_k = \gamma_k / \|q^{(k)}\|_2^2$
  $x^{(k+1)} = x^{(k)} + \alpha_k C^{-1} p^{(k)}$
  $r^{(k+1)} = r^{(k)} - \alpha_k q^{(k)}$
  $s^{(k+1)} = C^{-*}T^*r^{(k+1)}$
  $\gamma_{k+1} = \|s^{(k+1)}\|_2^2$
  $\beta_k = \gamma_{k+1}/\gamma_k$
  $p^{(k+1)} = s^{(k+1)} + \beta_k p^{(k)}$

The convergence rate of the PCGLS algorithm depends on the spectrum of the preconditioned matrix AC^{-1}, where C is an $n \times n$ nonsingular preconditioner matrix. Specifically, if the singular values of AC^{-1} are clustered around 1 then convergence will be rapid, cf. Axelsson and Barker [1]. The cost per iteration of PCGLS is dominated by matrix vector multiplies with A and A^*, and by linear system solves with C as a coefficient matrix. If A is an $m \times n$ Toeplitz matrix, then matrix vector multiplies with A and A^* can be accomplished in $O(m \log n)$ operations using the fast Fourier transform (FFT). Therefore, to make the PCGLS algorithm an efficient method for solving Toeplitz least squares problems, we must be able to construct a preconditioner matrix C such that (i) the singular values of AC^{-1} are clustered around 1, and (ii) the linear system with a coefficient matrix C can be easily solved. The construction of a preconditioner with these properties has been successfully done by the authors for an important class of Toeplitz matrices, arising in least squares problems, through the use of circulant approximations [8].

An $n \times n$ circulant matrix is a Toeplitz matrix that satisfies the additional property that each column (row) is a circular shift of the previous column (row). That is, the entries of C satisfy $c_{n-j} = c_{-j}$. An important property of circulant matrices is that they can be inverted in $O(n \log n)$ operations using the FFT, cf. Davis [14]. The circulant preconditioner described in [8] was obtained by partitioning the overdetermined matrix A into $n \times n$ submatrices, approximating the submatrices with circulant matrices, and then combining these to obtain a circulant approximation to A^*A.

In this paper we describe how to obtain an effecient circulant preconditioner for the solution of (1.1) by using the displacement structure of A^*A, without explicitely

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forming A^*A. An alternate displacement–based approach for the square case of A has been studied by Freund and Huckle [16].

In Section 2 we review some definitions and results on displacement representations of Toeplitz matrices. The development of a circulant preconditioner for overdetermined Toeplitz matrices based on the displacement representation of A^*A is introduced in Section 3. Displacement–based preconditioners have been shown to be very effective also in linear estimation and in adaptive filtering [24]. Additionally, Section 3 contains a detailed theoretical convergence analysis of the displacement preconditioner. In Section 4 some numerical results are reported, including comparisons with a block-based preconditioning scheme suggested in [8].

2. Displacement Structure. In this section we briefly review relevant definitions and results on displacement structure representation of a matrix. We introduce the $n \times n$ *lower shift* matrix Z, whose entries are zero everywhere except for 1's on the first subdiagonal. The *displacement operator* ∇ is defined by

$$\nabla A = A - ZAZ^*,$$

where ∇A is called the displacement of A, cf. Chun and Kailath [11]. Let $L(w)$ denote the $n \times n$ lower triangular Toeplitz matrix with first column the vector w. Using these definitions, the following lemma can be proved [12].

LEMMA 2.1. *An arbitrary $n \times n$ matrix A can be written in the form*

$$A = \sum_{i=1}^{\rho} L(u_i) L^*(v_i),$$

where $\rho = rank(\nabla A)$ and u_i and v_i are n–vectors.

The sum given in Lemma 2.1 above is called the *displacement representation* of the given matrix A and the scalar ρ is called the *displacement rank* of A. Square Toeplitz matrices and close to Toeplitz matrices have small displacement rank [12]. For example, if A is a Hermitian Toeplitz matrix, then

$$A = L(x_+)L(x_+)^* - L(x_-)L(x_-)^*,$$

where

$$x_\pm = [\frac{1}{2}(a_0 \pm 1), a_1, \cdots, a_{n-1}]^*.$$

To see this, we observe that $x_+ = x_- + e_1$, where $e_1 = [1, 0, \cdots, 0]^*$. Hence

$$x_+x_+^* - x_-x_-^* = e_1x_-^* + x_-e_1^* + e_1e_1^* = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & & & & \\ a_2 & & 0 & & \\ \vdots & & & & \\ a_{n-1} & & & & \end{bmatrix} = \nabla A.$$

If A is an $m \times n$, $m \geq n$, Toeplitz matrix, then A^*A is in general not a Toeplitz matrix. However the following well-known lemma indicates that A^*A does have a small displacement rank, $\rho \leq 4$, and provides a useful displacement representation for it.

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LEMMA 2.2. *Let A be an $m \times n$ Toeplitz matrix. Then a displacement representation of A^*A is*

$$A^*A = L(x_1)L(x_1)^* - L(x_2)L(x_2)^* + L(y_1)L(y_1)^* - L(y_2)L(y_2)^*,$$

where

$$x_1 = A^*Ae_1/||Ae_1||, \quad x_2 = ZZ^*x_1,$$

$$y_1 = [0, a_{-1}, a_{-2}, \cdots, a_{1-n}]^* \quad \textit{and} \quad y_2 = [0, a_{m-1}, a_{m-2}, \cdots, a_{m-n+1}]^*.$$

Proof. See [12], Lemma 2. □

Observe that $L(x_1) = L(x_2) + ||Ae_1||I$ and, therefore,

$$L(x_1)L(x_1)^* - L(x_2)L(x_2)^* = ||Ae_1||L(x_2) + ||Ae_1||L(x_2)^* + ||Ae_1||^2 I \equiv T,$$

where T is the Hermitian Toeplitz matrix with first column A^*Ae_1. Thus, A^*A can be expressed in the form

$$A^*A = T + L(y_1)L(y_1)^* - L(y_2)L(y_2)^*,$$

where T is Hermitian and Toeplitz and the $L(y_i)$ are lower triangular Toeplitz matrices.

3. Displacement Preconditioner. The idea of using circulant preconditioners in the PCG for solving square symmetric positive definite Toeplitz systems of equations was first proposed by Strang [25]. Since then, several other circulant preconditioning techniques have been proposed, see for instance T. Chan [10], R. Chan [6], Tyrtyshnikov [27], Ku and Kuo [19] and Huckle [18]. In particular, when A is an $n \times n$ Toeplitz matrix, T. Chan's circulant preconditioner (which we denote as $c(A)$) is defined to be the optimal circulant approximation to A in the Frobenius norm. That is, $c(A)$ is the circulant matrix which minimizes $||A - C||_F$ over all circulant matrices C. The diagonals c_j of $c(A)$ are given by

$$c_j = \begin{cases} \frac{n-j}{n} a_j + \frac{j}{n} a_{j-n}, & 0 \le j < n, \\ c_{n-j}, & -n < j < 0, \end{cases} \tag{3.1}$$

see [10] for details.

Circulant preconditioning has also been considered for solving least squares [8] and discrete ill-posed problems [17]. In [8], we constructed a circulant preconditioner for $m \times n$ matrices A, $m \ge n$, by partitioning A into $n \times n$ submatrices

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix},$$

approximating each A_i with $c(A_i)$, and then combining these to obtain a circulant approximation to A^*A. In this paper we take an alternate approach. Namely, we propose to use circulant approximations of the factors in the displacement representation to form a circulant approximation to A^*A.

It should be noted that in [8], as well as in our derivation below, any of the circulant approximations for $n \times n$ Toeplitz matrices can be used to derive circulant preconditioners for least squares problems. The T. Chan preconditioner $c(A)$ is defined for general square matrices A, not necessarily of Toeplitz form. We note that the operator c preserves the positive-definiteness of A. This is stated in the following Lemma due to Tyrtyshnikov [27].

Lemma 3.1. *If A is an $n \times n$ Hermitian matrix, then $c(A)$ is Hermitian. Moreover, we have*

$$\lambda_{\min}(A) \le \lambda_{\min}(c(A)) \le \lambda_{\max}(c(A)) \le \lambda_{\max}(A),$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and the smallest eigenvalues, respectively. In particular, if A is positive definite, then $c(A)$ is also positive definite.

We present our derivation and analysis of the displacement preconditioner for overdetermined least squares problems into two subsections. First we consider $n \times n$ Hermitian Toeplitz matrices A and show that the displacement preconditioner in this case is simply the T. Chan [10] approximation $c(A)$. We then use these results to derive the displacement preconditioner for the $m \times n$ case, and we provide a detailed convergence analysis. Our convergence analysis relies on the concept of generating functions for Toeplitz matrices. A function f defined on $[-\pi, \pi]$ is said to be a generating function of A if the diagonal entries, a_ℓ, of A are given by the Fourier coefficients of f, i.e.

$$a_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i\ell\theta} d\theta, \ \ \ell = 0, \pm 1, \pm 2, \cdots.$$

3.1. Hermitian Toeplitz Case. In this subsection, we consider the case where the matrix A is an n-by-n Hermitian Toeplitz matrix. We first recall that a displacement representation of A is given by

$$A = L(x_+)L(x_+)^* - L(x_-)L(x_-)^*, \tag{3.2}$$

where

$$x_\pm = [\frac{1}{2}(a_0 \pm 1), a_1, \cdots, a_{n-1}]^*. \tag{3.3}$$

Using (3.2), we define our preconditioner to be

$$C = c(L(x_+))c(L(x_+))^* - c(L(x_-))c(L(x_-)^*). \tag{3.4}$$

Clearly, C is a Hermitian circulant matrix.

Lemma 3.2. *Let A be a Hermitian Toeplitz matrix and C be the circulant approximation to A defined in (3.4). Then $C = c(A)$, the optimal circulant approximation to A.*

Proof. Let

$$x = [\frac{1}{2}a_0, a_1, \cdots, a_{n-1}]^*.$$

Then clearly $A = L(x) + L(x)^*$. We note also that

$$L(x_\pm) = L(x \pm \frac{1}{2}e_1) = L(x) \pm \frac{1}{2}L(e_1) = L(x) \pm \frac{1}{2}I,$$

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where I is the identity matrix. Thus by the linearity of the circulant operator c, we have

$$\begin{aligned} C &= c(L(x_+))c(L(x_+))^* - c(L(x_-))c(L(x_-))^* \\ &= [c(L(x) + \frac{1}{2}I)][c(L(x) + \frac{1}{2}I)]^* - [c(L(x) - \frac{1}{2}I)][c(L(x) - \frac{1}{2}I)]^* \\ &= c(L(x) + L(x)^*) = c(A). \end{aligned}$$

□

It follows from Lemma 3.1, that if A is positive definite, then so is $C = c(A)$. Using the convergence results on $c(A)$ presented in [9], we obtain the following result.

Corollary 3.3. *Suppose the generating function f of a Hermitian Toeplitz matrix A is a 2π-periodic continuous function. Then for all $\epsilon > 0$, there exist integers N and $M > 0$, such that when $n > N$, at most M eigenvalues of the matrix $C - A$ have absolute values larger than ϵ. If moreover f is positive, then the same property holds for the matrix $AC^{-1} - I$.*

It follows easily from the above Corollary that the conjugate gradient method, when applied to the preconditioned system AC^{-1}, converges superlinearly, see [9].

3.2. General Rectangular Toeplitz Case. In this subsection, we let A be an m-by-n rectangular Toeplitz matrix with $m \geq n$. Recall that the displacement representation of A^*A can be written as

$$A^*A = T + L(y_1)L(y_1)^* - L(y_2)L(y_2)^*, \tag{3.5}$$

where y_1, y_2 are given in Lemma 2, and T is a Hermitian Toeplitz matrix with

$$Te_1 \equiv \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{n-1} \end{bmatrix} = A^*Ae_1. \tag{3.6}$$

Substituting the displacement representation (3.2) for the symmetric Toeplitz matrix T in (3.5), we then have a displacement representation of A^*A:

$$A^*A = L(t_+)L(t_+)^* - L(t_-)L(t_-)^* + L(y_1)L(y_1)^* - L(y_2)L(y_2)^*, \tag{3.7}$$

where by (3.3)

$$t_\pm = [\frac{1}{2}(t_0 \pm 1), t_1, \cdots, t_{n-1}]^*.$$

Accordingly, we should define our preconditioner to be

$$c(L(t_+))c(L(t_+))^* - c(L(t_-))c(L(t_-))^* + c(L(y_1))c(L(y_1))^* - c(L(y_2))c(L(y_2))^*.$$

However, in Lemma 3.4 below we show that the contribution of the last term $L(y_2)L(y_2)^*$ in (3.7) is not significant as far as the conjugate gradient method is concerned, and we therefore will not approximate it by a circulant matrix. Thus our *displacement preconditioner* P is defined as follows

$$P = c(L(t_+))c(L(t_+))^* - c(L(t_-))c(L(t_-))^* + c(L(y_1))c(L(y_1))^*.$$

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According to Lemma 3.2, we see that

$$P = c(T) + c(L(y_1))c(L(y_1))^*. \tag{3.8}$$

In the following, we assume that the generating function f of A is in the Wiener class, i.e. the diagonals of A are absolutely summable:

$$\sum_{j=-\infty}^{\infty} |a_j| \le \gamma < \infty. \tag{3.9}$$

We note that Wiener class functions are 2π-periodic continuous functions. Under the Wiener class assumption, we will show that

$$P - A^*A = \{c(T) - T\} + \{c(L(y_1))c(L(y_1))^* - L(y_1)L(y_1)^*\} + L(y_2)L(y_2)^*$$

is the sum of a matrix of low rank and a matrix of small norm. For simplicity, in the following we denote by U_i Hermitian matrices with small rank and V_i Hermitian matrices with small norm. More precisely, given any $\epsilon > 0$, there exist integers N and $M > 0$, such that when n, the size of the matrices U_i and V_i, is larger than N, the rank of U_i is bounded by M and $||V_i||_2 < \epsilon$.

If the generating function of A is in the Wiener class, then so is the generating function of T. In fact,

$$||Te_1||_1 = ||A^*Ae_1||_1 \le ||A^*||_1 ||Ae_1||_1 \le \gamma^2 < \infty. \tag{3.10}$$

According to Corollary 3.3, we have

$$c(T) - T = U_1 + V_1. \tag{3.11}$$

Next we show that

$$c(L(y_1))c(L(y_1))^* - L(y_1)L(y_1)^* = U_2 + V_2. \tag{3.12}$$

The generating function of $L(y_1)$ is given by

$$g(\theta) = \sum_{j=-\infty}^{-1} a_j e^{ij\theta} \tag{3.13}$$

which is a function in the Wiener class. Equation (3.12) now follows by Lemma 5 of [8].

LEMMA 3.4.

$$L(y_2)L(y_2)^* = U_3 + V_3. \tag{3.14}$$

Proof. Since the sequence $\{a_j\}_{j=-\infty}^{\infty}$ is absolutely summable, for any given ϵ, we can choose $N > 0$ such that

$$\sum_{j>N} |a_j| < \epsilon.$$

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Partition $L(y_2)$ as $R_N + S_N$, where the first N columns of R_N are the first N columns of $L(y_2)$ with the remaining columns zero vectors. Then R_N is a matrix of rank N and

$$||S_N||_1 = \sum_{j=m-n+N+1}^{m-1} |a_j| \leq \sum_{j=N+1}^{m-1} |a_j| < \epsilon.$$

Thus

$$L(y_2)L(y_2)^* = (R_N + S_N)(R_N + S_N)^* = U_3 + V_3,$$

where

$$\text{rank}\ U_3 = \text{rank}(R_N S_N^* + S_N R_N^* + R_N R_N^*) \leq 2N$$

and

$$||V_3||_2 \leq ||S_N S_N^*||_2 \leq \epsilon^2.$$

□

Combining (3.11), (3.12) and (3.14), we see that

$$\begin{aligned} P - A^*A &= c(T) - T + c(L(y_1))c(L(y_1))^* - L(y_1)L(y_1)^* + L(y_2)L(y_2)^* \\ &= U_4 + V_4. \end{aligned} \tag{3.15}$$

Next we demonstrate that

$$P^{-1}(A^*A) - I = U_5 + V_5,$$

and we first show that $||P||_2$ and $||P^{-1}||_2$ are uniformly bounded. We begin with the bound for $||P||_2$.

LEMMA 3.5. *Let the generating function of the $m \times n$ Toeplitz matrix A be in the Wiener class, i.e. (3.9) holds. Then $||P||_2 \leq 6\gamma^2$ for all n.*

Proof. By (3.8) and Lemma 3.1,

$$||P||_2 \leq ||c(T)||_2 + ||c(L(y_1))c(L(y_1))^*||_2 \leq ||T||_2 + ||c(L(y_1))||_2^2.$$

It follows from (3.10) that

$$||T||_2 \leq ||T||_1 \leq 2||Te_1||_1 \leq 2\gamma^2.$$

On the other hand, using equation (9) in [8], we have

$$||c(L(y_1))||_2 \leq 2||g||_\infty,$$

where g is the generating function of $L(y_1)$ given in (3.13). Thus

$$||c(L(y_1))||_2 \leq 2|| \sum_{j=-\infty}^{-1} a_j e^{ij\theta} ||_\infty \leq 2\gamma.$$

□

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In order to show that $||P^{-1}||_2$ is uniformly bounded, we need the additional condition that

$$\min_{\theta\in[-\pi,\pi]} |f(\theta)| \geq \delta > 0. \tag{3.16}$$

LEMMA 3.6. *Let B be a square Toeplitz matrix with generating function in the Wiener class. Then*

$$\lim_{n\to\infty} \|c(B)c(B)^* - c(BB^*)\|_2 = 0.$$

Proof. The proof of the Lemma for Hermitian B is given in [7]. The case where B is not Hermitian but square can be proved similarly. ◻

LEMMA 3.7. *Let the generating function f of A be a Wiener class function that satisfies (3.16). Then $||P^{-1}||_2$ is uniformly bounded for n sufficiently large.*

Proof. Since the generating function g of $L(y_1)$ is in the Wiener class, it follows from Lemma 3.6, that given any $\epsilon > 0$,

$$c(L(y_1))c(L(y_1))^* - c(L(y_1)L(y_1)^*) = V_6,$$

where $||V_6||_2 < \epsilon$, provided that the size n of the matrix is sufficiently large. Hence

$$\begin{aligned} P &= c(T) + c(L(y_1))c(L(y_1))^* = c(T) + c(L(y_1)L(y_1)^*) + V_6 \\ &= c(T + L(y_1)L(y_1)^*) + V_6 = c(A^*A + L(y_2)L(y_2)^*) + V_6, \end{aligned}$$

where the last equality follows from (3.5). Write $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where A_1 is the n-by-n submatrix of A. The matrices A and A_1 have the same generating function f and $A^*A = A_1^*A_1 + A_2^*A_2$.

Since f by assumption is in the Wiener class it follows from Lemma 3.6 that,

$$c(A_1^*A_1) = c(A_1)^*c(A_1) + V_7,$$

where $||V_7||_2 \leq \epsilon$ if n is sufficiently large. Thus

$$\begin{aligned} P &= c(A^*A + L(y_2)L(y_2)^*) + V_6 \\ &= c(A_1^*A_1 + A_2^*A_2 + L(y_2)L(y_2)^*) + V_6 \\ &= c(A_1)^*c(A_1) + c(A_2^*A_2 + L(y_2)L(y_2)^*) + V_6 + V_7, \end{aligned} \tag{3.17}$$

Observe that

$$\{\lambda_{\min}[c(A_1)^*c(A_1)]\}^{-1} = ||[c(A_1)^*c(A_1)]^{-1}||_2 = ||c(A_1)^{-1}||_2^2 \leq 4||\frac{1}{f}||_\infty^2,$$

where the last inequality follows from equation (10) of [8]. Thus by (3.16),

$$\lambda_{\min}[c(A_1)^*c(A_1)] \geq \frac{\delta^2}{4}.$$

Since $A_2^*A_2 + L(y_2)L(y_2)^*$ is a positive semi-definite matrix, $c(A_2^*A_2 + L(y_2)L(y_2)^*)$ is also a positive semi-definite matrix. Thus we conclude from (3.17) that

$$\lambda_{\min}\{P\} \geq \lambda_{\min}[c(A_1)^*c(A_1)] - ||V_6||_2 - ||V_7||_2 \geq \frac{\delta^2}{4} - 2\epsilon.$$

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The Lemma follows by observing that ϵ is chosen arbitrarily and δ depends only on f and not on n. □

By combining (3.15), Lemmas 3.5 and 3.7, we see that if the generating function f of the $m \times n$ Toeplitz matrix A is a Wiener class function with no zeros on $[-\pi, \pi]$, then

$$P^{-1}(A^*A) - I = U + V,$$

where U is a low rank matrix and V is a small norm matrix. Thus the spectrum of the preconditioned matrix is clustered around one.

THEOREM 3.8. *Let the generating function f of the $m \times n$ Toeplitz matrix A be a Wiener class function with no zeros on $[-\pi, \pi]$. Then for all $\epsilon > 0$, there exist $N > 0$ and $M > 0$, such that for all $n > N$, at most M eigenvalues of the matrix*

$$P^{-1}(A^*A) - I$$

have absolute values larger than ϵ.

Proof. The proof is similar to the proof of Theorem 1 in [8]. □

From Theorem 1, we have the desired clustering result; namely, if the generating function f of the $m \times n$ Toeplitz matrix A is a Wiener class function with no zeros on $[-\pi, \pi]$, then the *singular values of the preconditioned matrix $AP^{-1/2}$ are clustered around* 1.

It can also be shown, in a manner similar to the derivation in §4 of [8], that if the condition number of A is of $O(n^\alpha)$, $\alpha > 0$, then the least squares conjugate gradient method converges in at most $O(\alpha \log n + 1)$ steps. Since each iteration requires $O(m \log n)$ operations using the FFT, it follows that the total complexity of the algorithm is only $O(\alpha m \log^2 n + m \log n)$.

When $\alpha = 0$, *i.e.*, $\kappa(A) = O(1)$, the number of iterations required for convergence is of $O(1)$. Hence the complexity of the algorithm reduces to $O(m \log n)$, for sufficiently large n. We remark that, in this case, one can show that the *method converges superlinearly* for the preconditioned least squares problem due to the clustering of the singular values for sufficiently large n (See [8] for details). In contrast, the method converges just linearly for the non-preconditioned case, as is illustrated by numerical examples in the next section.

4. Numerical Results. In this section we illustrate the effectiveness of the displacement preconditioner on some numerical examples. For each example we use the vector of all ones as the right hand side and the zero vector as the initial guess. The stopping criteria is $||s^{(j)}||_2/||s^{(0)}||_2 < 10^{-7}$, where $s^{(j)}$ is the normal equations residual after j iterations and is a by-product of the PCGLS computations. All computations were performed using Matlab 4.0 on an IBM RS/6000.

Throughout this section we denote a Toeplitz matrix with first column c and first row r as Toep(c, r). We present the number of iterations needed to converge using no preconditioner, the displacement preconditioner, and the preconditioner based on partitioning T as discussed in [8]. We denote these by "no prec", "disp prec" and "part prec", respectively.

The matrix in the first three examples satisfy the conditions of Theorem 1. We use T=Toep(c, r) as the coefficient matrix, where c and r are given as follows.

Example 1.

$$\begin{aligned} c(k) &= 1/k^2, \qquad k = 1, 2, \ldots, m \\ r(k) &= 1/k^2, \qquad k = 1, 2, \ldots, n. \end{aligned}$$

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Example 2.

$$\begin{aligned} c(k) &= e^{-0.1*k^2}, \qquad k = 1, 2, \ldots, m \\ r(k) &= e^{-0.1*k^2}, \qquad k = 1, 2, \ldots, n \end{aligned}$$

Example 3.

$$\begin{aligned} c(k) &= 1/\sqrt{k}, \qquad k = 1, 2, \ldots, m \\ r(k) &= 1/\sqrt{k}, \qquad k = 1, 2, \ldots, n \end{aligned}$$

Convergence results for these examples are reported in Table 1. Observe that the number of iterations needed for convergence for the preconditioned systems is essentially independent of the sizes of the matrices.

Table 4.1
Numbers of iterations for Examples 1, 2 and 3.

		Example 1.			Example 2.		
n	m	no prec	disp prec	part prec	no prec	disp prec	part prec
16	32	12	6	6	24	15	12
32	64	16	6	6	46	15	11
64	128	19	6	6	79	13	10
128	256	22	6	6	132	11	9
256	512	23	6	6	177	10	9

		Example 3.		
n	m	no prec	disp prec	part prec
64	128	59	8	8
64	256	61	6	8
64	512	62	6	8
64	1024	62	6	8
64	2048	64	8	8

Example 4. In this example we consider a convolution matrix, which is a 1-dimensional horizontal blurring function used in signal processing [2]. T =Toep(c, r) is defined by

$$\begin{array}{ll} c(k) = 1/2(w+1), & k = 1, 2, \ldots, w \\ c(k) = 0 & k = w+1, w+2, \ldots, m = n + w - 1 \\ r(1) = c(1) & \\ r(k) = 0, & k = 2, 3, \ldots, n \end{array}$$

The convergence results for this example are shown in Table 2.

These numerical results illustrate that the displacement preconditioner can significantly reduce the number of iterations needed for convergence of PCGLS for some examples. Moreover, as in Example 4, the displacement preconditioner scheme given here can be preferable to the partitioning approach to constructing circulant preconditioners discussed in [8]. However, we consider the main contribution in this paper to be the introduction of yet another preconditioner for possible use in solving Toeplitz least square problems. The choice of a (best) preconditioner is undoubtedly problem dependent.

TABLE 4.2
Numbers of iterations for Example 4.

n	m	no prec	disp prec	part prec
16	23	9	3	5
32	47	21	3	5
64	95	36	3	5
128	191	62	3	6
256	383	110	3	6

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