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AN ALGORITHM FOR NONHARMONIC SIGNAL ANALYSIS USING DIRICHLET SERIES ON CONVEX POLYGONS *

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Abstract. This article presents a new algorithm for nonharmonic signal analysis using Dirichlet series

$$f(z) = \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)}, \quad z \in D$$

on a convex polygon D as a generalization of Fourier series. Here L denotes a quasipolynomial whose set of zeros Λ generates a Riesz basis $\mathcal{E}(\Lambda) := \left\{ \frac{e^{\lambda z}}{L'(\lambda)} \right\}_{\lambda \in \Lambda}$ of the Smirnov space $E^2(D)$. The algorithm is based on a simple form of L and on numerical properties of the dual basis of $\mathcal{E}(\Lambda)$.

Key words. nonharmonic Fourier series, Dirichlet series, signal analysis, time series analysis.

AMS subject classifications. (2000) 42C15, 30B50, 37M10.

1. Introduction.

We consider functions $f \in L^2([-\pi, \pi])$ as they occur for example in physical systems such as electric circuits. Our aim is to decompose these functions into their basic oscillations, i.e., those components which are produced by the physical system or are connected directly with the system. Traditionally such functions are explored with Fourier methods by decomposing them into periodic oscillations of real frequencies

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int} \quad \text{in } L^2([-\pi, \pi]).$$

There is a wide class of functions coming from physical problems where an extraction of damped and upswinging oscillations is needed. These phenomena are modeled poorly with harmonic Fourier series of real frequencies as the following example shows.

1.1. Example: A damped oscillation.

We consider the real function $f_0 \in L^2([-\pi, \pi])$,

$$f_0(t) = 2e^{-0.3t} \cos(3t) = e^{i(3+0.3i)t} + e^{i(-3+0.3i)t},$$

(see Figure 1.1 (a)) composed of two oscillations with the complex frequencies $\lambda_1 = 3 + 0.3i$ and $\lambda_2 = -3 + 0.3i$ (see Figure 1.1 (b)). To analyze the function f_0 we calculate its complex Fourier transform

$$\mathcal{F}_c f_0(z) = \langle f_0, e^{iz\cdot} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_0(t) e^{-i\bar{z}t} dt, \quad z \in \mathbb{C}.$$

Figure 1.1 (c) shows a segment of the absolute value function $|\mathcal{F}_c f_0(z)|$. The complex Fourier transform is redundant, because $f_0 \in L^2([-\pi, \pi]) \subset L^1([-\pi, \pi])$ is uniquely represented by its harmonic Fourier series

$$f_0 = \sum_{n \in \mathbb{Z}} a_n e^{in\cdot} \quad \text{with } \{a_n\}_{n \in \mathbb{Z}} = \{\langle f_0, e^{in\cdot} \rangle\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}).$$

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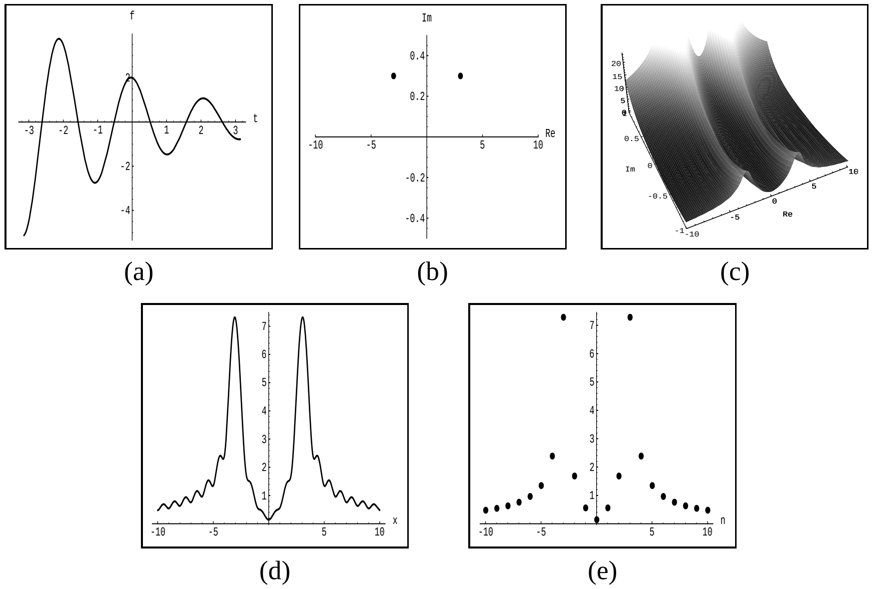

FIG. 1.1. *(a) The damped cosine function f_0 with complex frequencies (b), (c) the absolute value of its complex Fourier transform, (d) the absolute value of its real Fourier transform and (e) the absolute value of the coefficients of its harmonic Fourier expansion.*

Figure 1.1 (d) shows the absolute value of the coefficients $\{a_n\}_{n\in\mathbb{R}}$. We see clearly the bad localization of the spectrum. For a good approximation with a partial harmonic Fourier sum in $L^2([-\pi,\pi])$ we have to use many coefficients a_n, $n \in \mathbb{N}$ (see (e)). But even the canonical embedding $f \in L^2([-\pi,\pi]) \subset L^2([-A,A])$ with $A > \pi$ and $\operatorname{supp}(f) = [-\pi,\pi]$ and the choice of another harmonic Fourier basis in $L^2([-A,A])$ does not do better.

1.2. Assumptions. In many physical applications the interest lies in an interpretable basis decomposition. For the signal f in Example 1.1 the orthonormal Fourier basis decomposition gives no apparent information on the damping of f. To avoid this problem we start with the following general setting: Consider oscillations of the form

$$\rho e^{izt} = |\rho| e^{i\varphi} e^{ixt} e^{-yt}, \tag{1.1}$$

where $\rho = |\rho|e^{i\varphi} \in \mathbb{C}$, $\varphi \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$. The four parameters can be interpreted in physics in the following way: $|\rho|$ as amplitude, φ as phase, x as real frequency of the oscillation and y as damping.

Further, in most application problems we can assume preliminary knowledge of frequencies contained or expected in the physical system. So, let f contain the frequencies $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$.

Hence, our aim is to construct:

1. an adapted analyzing Riesz basis $\mathcal{E}(\Lambda) = \{e^{i\lambda t}\}_{\lambda\in\Lambda\subset\mathbb{C}}$ of complex exponentials
2. that contains the assumed frequencies $\lambda_1, \ldots, \lambda_N$ via the exponentials

$$e^{i\lambda_1 t}, \ldots, e^{i\lambda_N t} \in \mathcal{E}(\Lambda)$$

such that

3. we can represent f as a nonharmonic Fourier series

$$f = \sum_{\lambda\in\Lambda\subset\mathbb{C}} c_\lambda e^{i\lambda\cdot} \quad \text{in } L^2([-\pi,\pi]),$$

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where the coefficients $c_\lambda = \langle f, \psi_\lambda \rangle$ can be easily calculated numerically. (Here $\{\psi_\lambda\}_{\lambda\in\Lambda}$ denotes the dual basis corresponding to $\mathcal{E}(\Lambda)$.)

For the construction of an appropriate algorithm we consider a much more general setting, i.e., functions on convex polygons. We use the results known in the general case for the special case of the interval and show that this leads to an elegant algorithm in nonharmonic Fourier analysis also in the case of real frequencies.

2. Harmonic Fourier analysis. The traditional tools for signal analysis are the Fourier transform and the sampling theorem of Shannon, Whittaker and Kotel'nikov. The connection between these two well-known mappings gives the celebrated Paley–Wiener theorem. In this section we mention all three Theorems for comparison with Section 3.

THEOREM 2.1 (Fourier transform). *The Fourier transform*

$$\mathcal{F} : L^2([-\pi,\pi]) \to l^2(\mathbb{Z}),\ f \mapsto \{\widehat{f}(n)\}_{n\in\mathbb{Z}},$$

where

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\,dx$$

is an isometrical isomorphism. The inverse operator is given by

$$\mathcal{F}^{-1} : l^2(\mathbb{Z}) \to L^2([-\pi,\pi]),\ \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \mapsto f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\cdot}.$$

THEOREM 2.2 (Paley–Wiener). *Let B_π^2 denote the Bernstein space which consists of all entire functions of exponential type at most π that are square-integrable on the real axis. Then the mapping*

$$PW : L^2([-\pi,\pi]) \to B_\pi^2,\ f(t) \mapsto F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{e^{izt}}\,dt$$

is an isometrical isomorphism.

THEOREM 2.3 (Sampling theorem of Shannon, Whittaker and Kotel'nikov). *Any function F in the Bernstein space B_π^2 can be recovered from a countable number of sampling points by the isometrical isomorphism*

$$T : l^2(\mathbb{Z}) \to B_\pi^2,\ \{F(n)\}_{n\in\mathbb{Z}} \mapsto F(z),$$

where

$$F(z) = \sum_{n\in\mathbb{Z}} F(n)\frac{\sin(\pi(z-n))}{\pi(z-n)} = \sin(\pi z)\sum_{n\in\mathbb{Z}} F(n)\frac{(-1)^n}{\pi(z-n)}. \tag{2.1}$$

The series (2.1) is called cardinal series.

These three isometries form the commutative diagram in Figure 3.3 (a).

3. Generalizations: Irregular sampling and convex polygons. We give two generalizations of this commutative diagram: First, we look for sampling sequences other than $\mathbb{Z}$, and second, the interval $[-A, A]$ will be generalized to convex polygons $D \subset \mathbb{C}$. We will later use the results known for irregular sampling on convex polygons for the construction of our algorithm.

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3.1. From regular to irregular sampling. Looking at the cardinal series (2.1) the following questions arise: Is it necessary to sample entire functions at real and equidistant nodes? Can we also choose nodes in the complex plane? There are many references on irregular sampling, see for example [1, 2, 7, 8]. We restrict ourselves to sampling sequences that are zeros of sine type functions. They provide a Lagrange interpolation formula as in (2.1). As we will see later, for every finite set $\{\lambda_1, \ldots, \lambda_N\}$ of fixed complex numbers there exists a sine type function S with $S(\lambda_k) = 0$ for all $k = 1, \ldots, N$.

DEFINITION 3.1. *[4] An entire function $S(z)$ of exponential type A is called* sine type A function *if and only if, for some positive constants c, C and K we have*

$$c \cdot e^{A|\Im z|} \leq |S(z)| \leq C \cdot e^{A|\Im z|} \quad \textit{whenever } |\Im z| > K.$$

For sine type functions the following Theorem is valid:

THEOREM 3.2 (Levin, Golovin). *[4] Let $S(z)$ be a sine type A function with separated sequence of zeros $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, i.e., $\inf_{k\neq j} |\lambda_j - \lambda_k| > 0$. The zeros, λ_n, are numbered in increasing order of their real parts.*

Then the interpolation series $F(z) = \sum_{n\in\mathbb{Z}} F(\lambda_n) \frac{S(\lambda_n)}{S'(\lambda_n)(z-\lambda_n)}$ defines an isomorphism

$$T_S : l^2(\Lambda) \to B^2_\pi, \ \{F(\lambda_n)\}_{n\in\mathbb{Z}} \mapsto F(z).$$

Furthermore the system of exponential functions $\mathcal{E}(\Lambda) = \{e^{i\lambda_n \cdot}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2([-A, A])$.

We denote the analogue to the Fourier transform with respect to this Riesz basis by

$$\mathcal{F}_\Lambda : L^2([-\pi, \pi]) \to l^2(\Lambda), \ f \mapsto \{\langle f, \psi_{\lambda_n}\rangle\}_{n\in\mathbb{Z}},$$

where $\{\psi_{\lambda_n}\}_{n\in\mathbb{Z}}$ denotes the Riesz basis dual to $\mathcal{E}(\Lambda) = \{e^{i\lambda_n \cdot}\}_{n\in\mathbb{Z}}$. The inverse operator is given by

$$\mathcal{F}_\Lambda{}^{-1} : l^2(\Lambda) \to L^2([-\pi, \pi]), \ \{c_n\}_{n\in\mathbb{Z}} \mapsto f = \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n \cdot}.$$

Hence the commutative diagram of Figure 3.3 (a) can be extended to Figure 3.3 (b).

3.2. From the interval to convex polygons. In this section we examine the domain extension of the interval $[-\pi, \pi]$ to a convex polygon D. Let $D \subset \mathbb{C}$ be an open convex polygon that contains the origin of the complex plane. Let $a_1, \ldots, a_N$, $N \geq 3$, be the vertices of D and N_j normals on the sides of D. The angles between N_j and the positive real axis are denoted by θ_j (see Figure 3.1).

We define the following spaces generalizing $L^2([-\pi, \pi])$ respectively B^2_π to the domain D:

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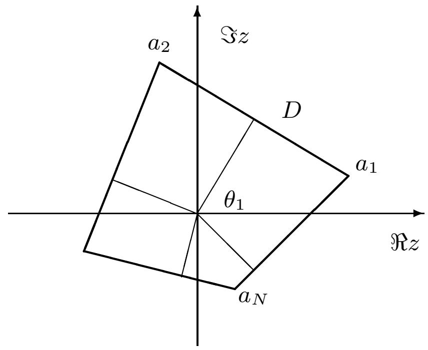

FIG. 3.1. *Prototype of the considered convex polygon.*

DEFINITION 3.3.

1. *Let $E_2(D)$ be the space of all functions f holomorphic in D and square-integrable on the boundary ∂D of D with norm*

$$\|f\|_{E_2(D)} = \left(\int_{\partial D} |f(z)|^2 \, |dz|\right)^{\frac{1}{2}}.$$

Then $E_2(D)$ is a separable Hilbert space and is called Smirnov space.

2. *[4] We denote by B_D^2 the space of all entire functions F of exponential type with finite norm*

$$\|F\|_{2,D} := \sup_{0 \le \theta \le 2\pi} \left\{ \int_0^\infty |F(re^{i\theta})|^2 e^{-2rk_D(\theta)} \, dr \right\}^{\frac{1}{2}}.$$

Here k_D is the supporting function $k_D(\theta) := \max_{z \in \overline{D}} (z \cdot e^{-i\theta})$ of D.

Let $\Pi_j(K)$ be a K-halfstrip in direction θ_j,

$$\Pi_j(K) := \{z \mid \Re(ze^{-i\theta_j}) > 0, \, |\Im(ze^{-i\theta_j})| < K\},$$

and $D_K := \bigcup_{j=1}^N \Pi_j(K)$ be the D_K-star (see Figure 3.2).

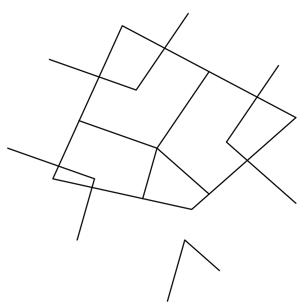

FIG. 3.2. *The D_K-star of a convex polygon.*

For convex polygons the definition of sine type functions is adapted as follows:

DEFINITION 3.4. *[4] An entire function $S(z)$ of exponential type is of* sine type *D if there exist positive constants c, C and K such that*

$$c \cdot e^{H_D(z)} \le |S(z)| \le C \cdot e^{H_D(z)}$$

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for all $z \in \mathbb{C} \setminus D_K$. *Here* $H_D(z) := |z| \cdot k_D(\arg z)$ *is the Minkovski function.*

With this generalization we obtain the following extension of Theorem 3.2:

THEOREM 3.5. *[4] Let* $S(z)$ *be of sine type* D *and* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ *be its separated zeros, i.e.,* $|\lambda_j - \lambda_k| > q > 0$ *for all* $\lambda_j \neq \lambda_k$. *Then the interpolation series*

$$F(z) \quad = \quad S(z) \sum_{\lambda\in\Lambda} F(\lambda) \frac{e^{H_D(\lambda)}}{S'(\lambda)(z-\lambda)}$$

converges absolutely, pointwise and in the L^2*-norm for every* $F \in B_D^2$. *It defines a topological isomorphism*

$$T_S : l^2(\Lambda) \to B_D^2, \{F(\lambda)\}_{\lambda\in\Lambda} \mapsto F(z).$$

The set $\mathcal{E}(\Lambda) := \{e^{\lambda \cdot} \cdot e^{-H_D(\lambda)}\}_{\lambda\in\Lambda}$ *forms a Riesz basis of* $E_2(D)$.

This leads to the following extensions of Theorems 2.1, 2.2 and 2.3:

1. Fourier type theorem.

$$\mathcal{F}_S \ : E_2(D) \to l^2(\Lambda), \ f \mapsto \{\kappa_f(\lambda)\}_{\lambda\in\Lambda}$$

with

$$\kappa_f(\lambda) = \int_{\partial D} f(z) \overline{B\left(\frac{e^{H(\lambda)}S(\cdot)}{S'(\lambda)(\cdot - \lambda)}\right)(z)}\, dz,$$

where B denotes the Borel-Transform of an entire function of exponential type for all $z \notin D^*$

$$BF(z) = \int_0^{\infty e^{i\theta}} F(t)e^{-zt}\, dt.$$

Here D^* denotes the complex conjugated polygon and the integration can be taken over each line from 0 to infinity, $\theta \in [0, 2\pi[$.

2. Paley–Wiener type theorem.

$$PW_D \ : E_2(D) \to B_D^2, \ f(t) \mapsto F(z) = \frac{1}{2\pi i}\int_{\partial D} f(t)\overline{e^{zt}}\, dt.$$

3. Sampling theorem.

$$T_S \ : l^2(\Lambda) \to B_D^2, \ \{F(\lambda)\}_{\lambda\in\Lambda} \mapsto F(z) = S(z)\sum_{\lambda\in\Lambda} F(\lambda)\frac{e^{H_D(\lambda)}}{S'(\lambda)(z-\lambda)}.$$

Figure 3.3 shows the extension of the three isomorphisms of §2 with sine type functions to irregular sampling in §3.1 and to convex polygons.

4. Quasipolynomials and Dirichlet series. We consider quasipolynomials

$$L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$$

as a special class of sine type D functions. For quasipolynomials the set of zeros Λ is approximately known. The following theorem characterizes the set of zeros. For an example see Figure 4.1.

THEOREM 4.1. *[3] The zeros* $\lambda_n^{(j)}$ *of the quasipolynomial* $L(z)$*—apart from finitely many denoted by* $\{\lambda_n\}_{n=1,\ldots,n_0}$ *and located in a neighborhood of the origin—are simple and*

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$$\begin{tikzcd} L^2([-A,A]) \arrow[r, "\mathcal{F}_{\mathbb{T}}"] \arrow[dr, "PW"'] & l^2(\mathbb{Z}) \arrow[d, "T"] \\ & B_A^2 \end{tikzcd} \qquad \begin{tikzcd} L^2([-A,A]) \arrow[r, "\mathcal{F}_{\Lambda}"] \arrow[dr, "PW"'] & l^2(\Lambda) \arrow[d, "T_S"] \\ & B_A^2 \end{tikzcd}$$

(a) (b)

$$\begin{tikzcd} E^2(D^*) \arrow[r, "\mathcal{F}_{\Lambda}"] \arrow[dr, "PW_D"'] & l^2(\Lambda) \arrow[d, "T_S"] \\ & B_D^2 \end{tikzcd}$$

(c)

FIG. 3.3. *(a) shows the commutative diagram consisting of the isometries in Theorems 2.1, 2.2 and 2.3. (b) With sine type functions the diagram can be extended to irregular sampling. (c) The domain is changed from an interval to a convex polygon.*

have the form

$$\lambda_n^{(j)} = \frac{2\pi n i}{a_{j+1} - a_j} + q_j e^{i\beta_j} + \delta_n^{(j)},$$

where $e^{q_j(a_{j+1}-a_j)e^{i\beta_j}} = -\frac{d_j}{d_{j+1}}$ *and* $|\delta_n^{(j)}| \le e^{-cn}$*. Hence, the set of zeros* Λ *of* $L(z)$ *consist of* $\{\lambda_n\}_{n=1,\ldots,n_0}$ *and* N *sequences* $\{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\ldots}$, $j = 1, \ldots, N$*:*

$$\Lambda = \{\lambda_n\}_{n=1,\ldots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\ldots} \right).$$

For the sequence of zeros Λ of the quasipolynomial L the family of complex exponentials $\mathcal{E}(\Lambda) = \{e^{\lambda\cdot}\}_{\lambda\in\Lambda}$ forms a Riesz basis of the Smirnov space $E_2(D)$. Furthermore, there are explicit formulas for the dual basis:

$$\psi_\lambda = -e^{\lambda\cdot} \int_0^t \sum_{k=1}^{N} d_k \frac{1}{\xi - a_k} \cdot e^{\lambda\cdot}\, d\xi, \quad \lambda \in \Lambda.$$

Hence the Fourier type theorem of subsection 3.2 now has the following explicit form:

$$\mathcal{F}_L \;:\; E_2(D) \to l^2(\Lambda),\; f \mapsto \left\{ \frac{\kappa_f(\lambda)}{L'(\lambda)} \right\}_{\lambda\in\Lambda}$$

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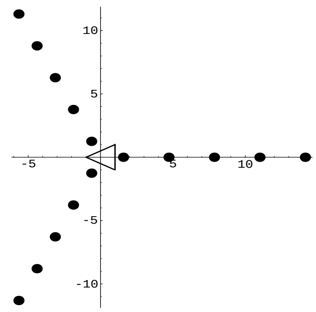

FIG. 4.1. *The quasipolynomial $L(z) = \sum_{k=1}^{3} e^{a_k z}$ to the triangle with vertices $a_1 = 1 + i$, $a_2 = -1$ and $a_3 = 1 - i$ is a sine type function with zeros located in small neighborhoods of the points denoted.*

with

$$\kappa_f(\lambda) = \sum_{k=1}^{N} d_k \, e^{a_k \lambda} \int_{a_j}^{a_k} f(z) e^{-\lambda z} dz,$$

where a_j is an arbitrary, but fixed vertex of D. The inverse mapping

$$\mathcal{F}_L^{-1} \; : \; l^2(\Lambda) \to E_2(D), \; \left\{ \frac{\kappa_f(\lambda)}{L'(\lambda)} \right\}_{\lambda \in \Lambda} \mapsto f(z)$$

leads to so-called Dirichlet series

$$f(z) = \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)}.$$

5. Generalized Jackson theorems. In the previous three paragraphs we saw that Dirichlet series are a special case of the extension of harmonic analysis on the interval to nonharmonic analysis on convex polygons. Hence it is natural to ask if Jackson type theorems are valid for Dirichlet series. This question can be answered positively for the norm of uniform convergence as well as for the L^p-norm and for first moduli of smoothness. Jackson type theorems are also true for moduli of higher order in the uniform norm. For the L^p-norm, $1 < p < \infty$, this is an open problem. Here we will restrict ourselves to first moduli and the uniform norm.

DEFINITION 5.1. *Let $AC(\overline{D})$ denote the space of all $f \in C(\partial D)$, that are holomorphic in D. Endowed with the norm*

$$\|f\|_{AC(\overline{D})} := \sup_{z \in \partial D} |f(z)|,$$

$AC(\overline{D})$ becomes a Banach space. Let ω be a first modulus of continuity, i.e., $\omega(h)$ is defined for $h > 0$, nondecreasing, semiadditive and $\lim_{h \to 0+} \omega(h) = 0$. We denote by $AW^r H^\omega(\overline{D})$ the class of all $f \in AC(\overline{D})$ with the condition that for all $z_1, z_2 \in \mathbb{C}$ the following estimate is true:

$$|f^{(r)}(z_1) - f^{(r)}(z_2)| \leq c \cdot \omega(|z_1 - z_2|).$$

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Let $n = (n(1), \ldots, n(N)) \in \mathbb{N}^N$ and $r \in \mathbb{N}_0$. Then

$$P_{n,r}(z) = \sum_{m=1}^{m_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=m(j)}^{n'_j} (1 - \xi_{j,m}^{r+1}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}$$

denotes the Jackson (n,r) quasipolynomial. Here $n'_j = 2\left\lfloor \frac{n_j - 1}{2} \right\rfloor$ and $\xi_{j,m} = 1 - J_{j,m}$ with

$$\frac{3}{2n_j(2n_j^2+1)} \left(\frac{\sin(n_j t/2)}{\sin(t/2)} \right)^4 = \frac{J_{j,0}}{2} + \sum_{m=1}^{n'_j} J_{j,m} \cos(mt).$$

The following Jackson type theorem is due to Y. I. Mel'nik:

THEOREM 5.2 (Mel'nik). *[6] Let the modulus of continuity ω satisfy the Zygmund condition*

$$\int_0^h \frac{\omega(t)}{t} dt + h \int_h^{2\pi} \frac{\omega(t)}{t^2}\, dt \le A \cdot \omega(h)$$

for some positive constant A. Then (i) and (ii) are equivalent:

(i) $f \in AW^r H^\omega(\overline{D})$ satisfies the conditions

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0, \quad 0 \le s \le r.$$

(ii) There is a sequence of Jackson (n,r) quasipolynomials $\{P_{n,r}\}_n$ where $n = (n(1), \ldots, n(N)) \in \mathbb{Z}^n$, and a positive constant A_1 with

$$\|f - P_{n,r}\|_{AC(\overline{D})} \le A_1 \cdot \sum_{k=1}^{N} \frac{1}{(n(k))^r} \cdot \omega\left(\frac{1}{n(k)} \right).$$

In the proof, the connection between Leont'ev coefficients $\kappa_f(\lambda)$ and Fourier coefficients plays an essential role.

LEMMA 5.3 (Mel'nik). *[5] Let $f \in AH^\omega(\overline{D})$, $\frac{\omega(t)}{t}$ be integrable on $[0,\delta]$, $\delta > 0$, and*

$$\sum_{k=1}^{N} d_k f(a_k) = 0.$$

Then the coefficients

$$\kappa_f(\lambda_n^{(j)}) = \sum_{k=1}^{N} d_k\, e^{a_k \lambda} \int_{a_j}^{a_k} f(z) e^{-\lambda z} dz$$

of the Dirichlet series

$$f(z) = \sum_{\lambda_n^{(j)} \in \Lambda} \kappa_f(\lambda_n^{(j)}) \frac{e^{\lambda_n^{(j)} z}}{L'(\lambda_n^{(j)})}$$

are the Fourier coefficients of a continuous function $F_j(t) \in H^\Omega([0, 2\pi])$, i.e.,

$$|F_j(t_1) - F_j(t_2)| \le C \cdot \Omega(|t_1 - t_2|) \quad \textit{for all } t_1, t_2 \in [0, 2\pi],$$

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where C is a positive constant and

$$\Omega(h) \leq A \cdot \left\{ \int_0^h \frac{\omega(t)}{t} dt + h \int_h^{2\pi} \frac{\omega(t)}{t^2} dt \right\}.$$

6. Application: A new algorithm for signal analysis. Let $f \in L^2([-\pi, \pi])$. We assume that f contains the frequencies $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. With the results of the previous section we now have all we need to construct an adapted Riesz basis $\mathcal{E}(\Lambda) = \{e^{i\lambda \cdot}\}_{\lambda \in \Lambda}$ of complex exponentials with $e^{i\lambda_1 \cdot}, \dots, e^{i\lambda_N \cdot} \in \mathcal{E}(\Lambda)$, that fulfills all requirements of §1.2. We consider the following algorithm for the calculation of the appropriate Riesz basis and the corresponding coefficients:

ALGORITHM 1.

1. *Consider the sine type quasipolynomial*

$$L(z) = e^{i\pi z} + e^{-i\pi z} + \sum_{k=3}^{N+2} d_k e^{a_k z}$$

with $d_1 = d_2 = 1$, $d_k \in \mathbb{C}$. Here $a_1 = i\pi$, $a_2 = -i\pi$ and $a_k \in]-i\pi, i\pi[$ have to be chosen such that the zeros Λ of $L(z)$ are separated and with $-ia_1 > -ia_3 > -ia_4 > \dots > -ia_{N+2} > -ia_2$.

2. *Calculate d_k, such that $L(\lambda_m) = 0$ for $m = 1, \dots, N$, i.e., solve the system of N linear equations*

$$\begin{pmatrix} e^{a_3\lambda_1} & \dots & e^{a_{N+2}\lambda_1} \\ \vdots & & \vdots \\ e^{a_3\lambda_N} & \dots & e^{a_{N+2}\lambda_N} \end{pmatrix} \begin{pmatrix} d_3 \\ \vdots \\ d_{N+2} \end{pmatrix} = - \begin{pmatrix} e^{i\pi\lambda_1} + e^{-i\pi\lambda_1} \\ \vdots \\ e^{i\pi\lambda_N} + e^{-i\pi\lambda_N} \end{pmatrix}.$$

This can by done by some form of Gaussian elimination.

3. *Calculate the zeros $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of $L(z)$ with the help of Theorem 4.1. We interpret the points $a_1, \dots, a_{N+2}$ as vertices of a degenerate convex polygon. For this view, the points $a_3, \dots, a_{N+2}$ are counted twice and moved with respect to some small $\varepsilon > 0$. Hence we get a polygon with vertices in the order $a_1, a_3 - \varepsilon, \dots, a_{N+2} - \varepsilon, a_2, a_{N+2} + \varepsilon, \dots, a_3 + \varepsilon, a_1$ (see Figure 6.1 for an example).*

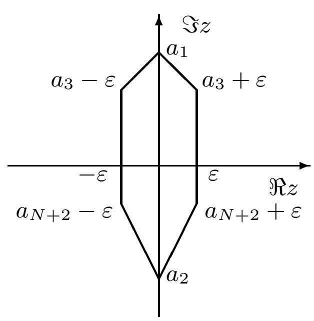

FIG. 6.1. *The construction of a convex polygon from an interval.*

For this polygon Theorem 4.1 gives the zeros of the corresponding quasipolynomial $L_\varepsilon(z) = e^{i\pi z} + e^{-i\pi z} + \sum_{k=3}^{N+2} d_k(e^{(a_k+\varepsilon)z} + e^{(a_k-\varepsilon)z})$. For ε small enough they are good starting points for standard zero seeking algorithms, for example the damped Newton's method. If the zeros Λ of L are not simple, change a_k, $k = 3, \dots, N+2$, in step 1.

4. *Calculate the coefficients*

$$\kappa_f(\lambda) = \sum_{k=1}^{N} d_k e^{a_k \lambda} \int_{a_j}^{a_k} f(\eta) e^{-\lambda\eta} d\eta,$$

where a_j can be arbitrarily chosen from the set of vertices. The calculation is numerically easy, because the formula just contains a finite sum and integrals over a finite interval. For example, the integration can be computed by an adaptive Gaussian quadrature.

5. *As result we get the nonharmonic Fourier representation*

$$f(z) = \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)} \quad \textit{in } L^2([-\pi, \pi]).$$

The advantages of this new construction are that preliminary knowledge can be used in the construction of the quasipolynomial L. For an adequate choice of the vertices a_k we decompose with respect to a Riesz basis of complex exponentials. The special form of quasipolynomials gives an explicit formula for the series' coefficients. This formula can be easily evaluated numerically.

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