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ASYMPTOTICS FOR QUADRATIC HERMITE-PADÉ POLYNOMIALS ASSOCIATED WITH THE EXPONENTIAL FUNCTION *

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Abstract. The asymptotic behavior of quadratic Hermite-Padé polynomials $p_n, q_n, r_n \in \mathcal{P}_n$ of type I and $\mathfrak{p}_n$, $\mathfrak{q}_n$, $\mathfrak{r}_n \in \mathcal{P}_{2n}$ of type II associated with the exponential function are studied. In the introduction the background of the definition of Hermite-Padé polynomials is reviewed. The quadratic Hermite-Padé polynomials $p_n, q_n, r_n \in \mathcal{P}_n$ of type I are defined by the relation

$$p_n(z) + q_n(z)e^z + r_n(z)e^{2z} \;=\; O(z^{3n+2}) \quad \text{as} \quad z \to 0,$$

and the polynomials $\mathfrak{p}_n$, $\mathfrak{q}_n$, $\mathfrak{r}_n \in \mathcal{P}_{2n}$ of type II by the two relations

$$\begin{aligned} \mathfrak{p}_n(z)e^z - \mathfrak{q}_n(z) \;&=\; O(z^{3n+1}) \quad \text{as} \quad z \to 0, \\ \mathfrak{p}_n(z)e^{2z} - \mathfrak{r}_n(z) \;&=\; O(z^{3n+1}) \quad \text{as} \quad z \to 0. \end{aligned}$$

Analytic descriptions are given for the arcs, on which the contracted zeros of both sets of the polynomials $\{p_n, q_n, r_n\}$ and $\{\mathfrak{p}_n, \mathfrak{q}_n, \mathfrak{r}_n\}$ cluster as $n \to \infty$. Analytic expressions are also given for the density functions of the asymptotic distributions of these zeros.

The description is based on an algebraic function of third degree and a harmonic function defined on the Riemann surface, which is associated with the algebraic function. The existence and basic properties of the asymptotic distributions of the zeros and the arcs on which these distributions live are proved, the asymptotic relations themselves are only conjectured. Numerical calculations are presented, which demonstrate the plausibility of these conjectures.

Key words. Quadratic Hermite-Padé polynomials of type I and type II, the exponential function, German and Latin polynomials, Hermite-Padé approximants.

AMS subject classifications. 41A21, 30E10.

1. Introduction and Review.

The first part of the talk has review character, the polynomials considered in the second part of the talk are special cases of general Hermite-Padé polynomials, which can be seen as a generalization of Padé polynomials, and therefore also as generalizations of orthogonal polynomials. The approximants associated with the Hermite-Padé polynomials are generalizations of the concepts of Padé approximants and continued fractions. In order to motivate the definition of Hermite-Padé polynomials we start by repeating the definition of diagonal Padé approximants and their associated orthogonal polynomials.

1.1. Padé Approximants.

Let the function f be analytic at the origin. Then there exist polynomials p_n, $q_n \in \mathcal{P}_n$ such that

$$p_n(z) - q_n(z)f(z) \;=\; O(z^{2n+1}) \quad \text{as} \quad z \to 0 \tag{1.1}$$

with $O(\cdot)$ denoting Landau's big 'oh'. The rational function $[n/n]_f := p_n/q_n$ is uniquely determined by (1.1), and it is called the diagonal Padé approximant of degree (n, n) to the function f. The denominator polynomial q_n of $[n/n]_f$ can be characterized by orthogonality. Let $Q_n(z) := z^n q_n(1/z)$ denote the reversed polynomial of q_n. Then the polynomial q_n satisfies relation (1.1) (together with an appropriate choice of p_n) if and only if the reversed polynomial Q_n satisfies the orthogonality relation

$$\oint_C \zeta^l Q_n(\zeta) f(1/\zeta) d\zeta = 0, \qquad l = 0, \dots, n-1, \tag{1.2}$$

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where C is an integration path around infinity. In case of Markov, Stieltjes, or Hamburger functions, relation (1.2) transforms into an orthogonality relation defined by a positive measure on $\mathbb{R}$ (cf. [3], Chapter 5.3, [17], Chapter 2, or [18], Kapitel 9 & 10).

We have $f \equiv [n/n]_f$ for n sufficiently large if and only if f is a rational function. This assertion is known as Kronecker's Theorem. In a letter to Jacobi, Hermite raised the question whether something similar, but more general than continued fractions (more general than Padé approximants, we would say today) could be defined so that the algebraic character of a function with degree $m > 1$ could be detected in the same way as rational functions can be detected by continued fractions (or Padé approximants) via Kronecker's Theorem. (Note that rational functions are algebraic of degree 1.) A discussion of the correspondence between Hermite and Jacobi can be found in [5], page 1-11, where one also finds further references about this topic.

1.2. Hermite-Padé Polynomials. The question posed by Hermite led to the introduction of what is now called the Jacobi-Perron algorithm (cf. [11], [19], [17], Chapter 4.5) and further to the definition of Hermite-Padé polynomials and their associated approximants. Here, we will not discuss the Jacobi-Perron algorithm, which is a generalization of the continued fraction approach, and instead concentrate on the Hermite-Padé polynomials and their associated approximants.

Let $\mathfrak{f} = (f_0, \ldots, f_m)$, $m \geq 1$, be a system of functions, which are analytic in a neighborhood of the origin.

DEFINITION 1.1. ***Hermite-Padé Polynomials of Type I*** *(Latin polynomials in K. Mahler's terminology [16]): For any multi-index $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ there exists a vector of polynomials $(p_0, \ldots, p_m) \in \mathcal{P}^*_{n_0-1} \times \mathcal{P}_{n_1-1} \times \ldots \times \mathcal{P}_{n_m-1}$ such that*

$$\sum_{j=0}^{m} p_j(z) f_j(z) \;=\; O(z^{|n|-1}) \quad \text{as} \quad z \to 0, \tag{1.3}$$

where $|n| := n_0 + \ldots + n_m$ and $\mathcal{P}^*_k := \{\, p \in \mathcal{P}_k \,|\, p \text{ monic}, p \not\equiv 0 \,\}$. The vector $(p_0, \ldots, p_m)$ is called Hermite-Padé form of type I, and its elements are the Hermite-Padé polynomials of type I.

DEFINITION 1.2. ***Hermite-Padé Polynomials of Type II*** *(German polynomials in K. Mahler's terminology [16]): For any multi-index $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ there exists a vector of polynomials $(\mathfrak{p}_0, \ldots, \mathfrak{p}_m) \in \mathcal{P}^*_{N_0} \times \mathcal{P}_{N_1} \times \ldots \times \mathcal{P}_{N_m}$ with $N_j := |n| - n_j$, $j = 0, \ldots, m$, such that*

$$\mathfrak{p}_i(z) f_j(z) - \mathfrak{p}_j(z) f_i(z) \;=\; O(z^{|n|+1}) \quad \text{as} \quad z \to 0, \tag{1.4}$$

for $i, j = 0, \ldots, m$, $i \neq j$. The vector $(\mathfrak{p}_0, \ldots, \mathfrak{p}_m)$ is called Hermite-Padé form of type II, and its elements are the Hermite-Padé polynomials of type II.

Remarks: (1) The assumption $p_0 \in \mathcal{P}^*_{n_0-1}$ and $\mathfrak{p}_0 \in \mathcal{P}^*_{N_0}$ implies a normalization of the whole form $(p_0, \ldots, p_m)$ and $(\mathfrak{p}_0, \ldots, \mathfrak{p}_m)$, respectively. There may exist situations in which a normalization by the first component is not possible; however, there always exists one of the $m+1$ components by which a normalization is possible.

(2) In general the Hermite-Padé polynomials are not unique. Their existence is immediate since both relations (1.3) and (1.4) can be rewritten as homogeneous linear systems of equations for the coefficients of the polynomials.

(3) In (1.4) there are $(m+1)m/2$ formally different relations. However, at most m of these relations are linearly independent.

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1.3. Hermite-Padé Approximants. If one takes $m = 1$, $f_0 \equiv -1$, and $f_1 = f$, then relation (1.3) reduces to (1.1), and by taking $m = 1$, $f_0 \equiv 1$, and $f_1 = f$, in relation (1.4) one again gets relation (1.1). Hence, the concept of Padé polynomials $p_n, q_n \in \mathcal{P}_n$ defined by (1.1) is a special case of both types of Hermite-Padé polynomials.

If $f_0(0) \neq 0$, then one can assume without loss of generality that $f_0 \equiv 1$, and under this assumption we deduce from (1.4) that

$$\mathfrak{p}_0(z)f_j(z) - \mathfrak{p}_j(z) \;=\; O(z^{|n|+1}) \quad \text{as} \quad z \to 0 \quad \text{for} \quad j = 1, \ldots, m. \tag{1.5}$$

DEFINITION 1.3. ***Hermite-Padé Simultaneous Rational Approximants****: For a given multi-index $n \in \mathbb{N}^{m+1}$ let $\mathfrak{p}_0, \ldots, \mathfrak{p}_m$ be the Hermite-Padé polynomials of type II define by (1.5). Then the vector of rational functions*

$$\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_0}(z), \ldots, \frac{\mathfrak{p}_m}{\mathfrak{p}_0}(z)\right) \tag{1.6}$$

,with common denominator polynomial $\mathfrak{p}_0$ is called the (Hermite-Padé) simultaneous rational approximant to the (reduced) system of functions $\mathfrak{f}_{red} = (f_1, \ldots, f_m)$. For $m = 1$ we have the Padé approximants to f_1 with numerator and denominator degrees (n_0, n_1) as special case of (1.6) .

Besides the simultaneous rational approximants there exists a second type of approximants: the algebraic Hermite-Padé approximants. They are defined with the help of Hermite-Padé polynomials of type I.

Let f be a function analytic at the origin, and define the system of functions $\mathfrak{f}$ as

$$\mathfrak{f} = (f_0, \ldots, f_m) := (1, f, \ldots, f^m). \tag{1.7}$$

DEFINITION 1.4. ***Algebraic Hermite-Padé Approximants****: For a given multi-index $n \in \mathbb{N}^{m+1}$ let $p_0, \ldots, p_m \in \mathcal{P}^*_{n_0-1} \times \ldots \times \mathcal{P}_{n_m-1}$ be the Hermite-Padé polynomials of type I defined by (1.3) with the special choice of (1.7). Let the algebraic function $y = y(z)$ be defined by*

$$\sum_{j=0}^{m} p_j(z)y(z)^j \;\equiv\; 0. \tag{1.8}$$

From the m branches of y we select the branch $y = y_n$ which has the highest contact with f at the origin. This branch y_n is the algebraic Hermite-Padé approximant to f associated with the multi-index n.

Again, it is immediate that, for $m = 1$ Definition 1.4 leads to the Padé approximants $[n_0 - 1/n_1 - 1]_f$ introduced after (1.1). For $m > 1$ the two types of Hermite-Padé approximants introduced in Definition 1.3 and 1.4 split up in two different directions: In the first case we continue to use rational functions as approximants, but we now approximate simultaneously several functions; in the second case only one function f is approximated, but now by an algebraic function of $m-$th degree so that in principle again m branches of the function f can be approximated simultaneously by the m branches of the approximant. A survey over asymptotics of Hermite-Padé polynomials together with related results about the convergence of Hermite-Padé approximants can be found in [2].

1.4. Orthogonality. As in (1.2), and also in case of Hermite-Padé polynomials, one can express the defining relations (1.3), (1.4) or (1.5) by orthogonality in an equivalent way. Here, we discuss the orthogonality relations only for the diagonal case, i.e., we assume that all multi-indices $n \in \mathbb{N}^{m+1}$ are of the form $n = (k, \ldots, k)$ with $k \in \mathbb{N}$. The reversed polynomials of $p_0, \ldots, p_m$ and $\mathfrak{p}_0, \ldots, \mathfrak{p}_m$ are defined by

$$P_j(z) := z^{k-1} p_j(1/z) \quad \text{and} \quad \mathfrak{P}_j(z) := z^{mk} \mathfrak{p}_j(1/z), \quad j = 1, \ldots, m. \tag{1.9}$$

Further, we assume that $f_0 \equiv 1$. Under these assumptions a vector of polynomials $(p_1, \ldots, p_m) \in \mathcal{P}_{k-1} \times \ldots \times \mathcal{P}_{k-1}$ satisfies relation (1.3) together with an appropriately chosen polynomial $p_0 \in \mathcal{P}^*_{k-1}$ if and only if

$$\oint_C \zeta^l \sum_{j=1}^{m} P_j(\zeta) f_j(1/\zeta) d\zeta = 0 \quad \text{for} \quad l = 0, \ldots, mk-1. \tag{1.10}$$

A polynomial $\mathfrak{p}_0 \in \mathcal{P}^*_{mk}$ satisfies relation (1.4) together with an appropriately chosen set of polynomials $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathcal{P}_{mk}$ if and only if

$$\oint_C \zeta_j^l \mathfrak{P}_0(\zeta) f_j(1/\zeta) d\zeta = 0 \quad \text{for} \quad l = 0, \ldots, k-1, \quad j = 1, \ldots, m. \tag{1.11}$$

In both integrals C is an integration path encircling infinity. There also exist orthogonality relations analogous to (1.11) for the other polynomials $\mathfrak{P}_j$, $j = 1, \ldots, m$, of type II.

If the functions $f_1, \ldots, f_m$ are of Markov, Stieltjes, or Hamburger type, then the m orthogonality relations in (1.11) take a more conventional form (cf. [17], Chapter 4). The polynomial $\mathfrak{P}_0$ in (1.11) is called multiple orthogonal since it satisfies simultaneously m different orthogonality relations, each one with a partial degree $n_j - 1$. The total number of restrictions provided by the multiple orthogonality relation (1.11) is $|n| - n_0 = n_1 + \ldots + n_m$, which is the maximally possible degree of $\mathfrak{P}_0$. The theory of multiple orthogonality is still in an early stage of development. Nevertheless, there exists already a considerable amount of knowledge and special results. Surveys about the state of art in this field can be found in [17], Chapter 4, [1], and [23].

1.5. Mahler Relations. Up to this point one may have the impression that the Definitions 1.1 and 1.2 of both types of Hermite-Padé polynomials are rather independent if $m > 1$. But this is not so. Under certain 'normality' assumptions one can calculate one type of polynomials from the other one. The situation is especially simple if the system of functions $\mathfrak{f}$ is perfect.

DEFINITION 1.5. ***Perfect Systems:*** *A System of functions $\mathfrak{f} = (f_0, \ldots, f_m)$ is called perfect if for all multi-indices $n \in \mathbb{N}^{m+1}$ the left-hand side of (1.3) starts with the term $Az^{|n|-1}$ and $A \neq 0$.*

The system is called weakly perfect if this property holds true only for close-to-diagonal multi-indices $n \in \mathbb{N}^{m+1}$. More precisely, it is weakly perfect if it holds true for each multi-index $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ with the property that there exists $k \in \mathbb{N}$ with $0 \leq n_j - k \leq 1$ for all $j = 0, \ldots, m$.

The perfectness of special systems has been studied in [16], [7], and [12]. Weak perfectness has been established for Angelesco and Nikishin systems (cf. [17], Chapter 4).

In order to define the (diagonal) Mahler relations, we introduce for each $k \in \mathbb{N}$ two sets of multi-indices $n(k)_0, \ldots, n(k)_m \in \mathbb{N}^{m+1}$ and $\mathfrak{n}(k)_0, \ldots, \mathfrak{n}(k)_m \in \mathbb{N}^{m+1}$ with $n(k)_i =$

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$(n_{i0}, \ldots, n_{im})$, $\mathfrak{n}(k)_i = (\mathfrak{n}_{i0}, \ldots, \mathfrak{n}_{im})$, $i = 0, \ldots, m$. The $n_{ij} = n_{ij}(k)$ and $\mathfrak{n}_{ij} = \mathfrak{n}_{ij}(k)$ are defined by

$$n_{ij}(k) := k + \delta_{ij}, \qquad \mathfrak{n}_{ij}(k) := k - \delta_{ij}, \quad i, j = 0, \ldots, m, \quad k \in \mathbb{N}, \tag{1.12}$$

where δ_{ij} denotes Kronecker's symbol. We have $|n(k)_i| = (m+1)k + 1$ and $|\mathfrak{n}(k)_i| = (m+1)k - 1$ for $i = 0, \ldots, m$.

For the $m+1$ multi-indices $n(k)_i \in \mathbb{N}^{m+1}$, $i = 0, \ldots, m$, and the system $\mathfrak{f} = (f_0, \ldots, f_m)$ the $m+1$ vectors of Hermite-Padé polynomials of type I are denoted by

$$(p_{i0}, \ldots, p_{im}) \in \mathcal{P}_{k-1} \times \ldots \times \mathcal{P}^*_k \times \ldots \times \mathcal{P}_{k-1}, \quad i = 0, \ldots, m. \tag{1.13}$$

The assumptions in (1.13) imply a normalization different from that used in Definition 1.1. Here, the $i-$th component is used for normalizing the vector $(p_{i0}, \ldots, p_{im})$. If the system $\mathfrak{f}$ is weakly perfect, then this normalization is always possible.

For the multi-indices $\mathfrak{n}(k)_i \in \mathbb{N}^{m+1}$, $i = 0, \ldots, m$, and the system $\mathfrak{f}$, the vectors of Hermite-Padé polynomials of type II are denoted by

$$(\mathfrak{p}_{i0}, \ldots, \mathfrak{p}_{im}) \in \mathcal{P}_{mk-1} \times \ldots \times \mathcal{P}^*_{mk} \times \ldots \times \mathcal{P}_{mk-1}, \quad i = 0, \ldots, m. \tag{1.14}$$

Again, the normalization implied by (1.14) is different from that used in Definition 1.2. The vector of polynomials in (1.13) and (1.14) are put together in two matrices:

$$P := \begin{pmatrix} p_{00} & \cdots & p_{0m} \\ \vdots & & \vdots \\ p_{m0} & \cdots & p_{mm} \end{pmatrix}, \qquad \mathfrak{P} := \begin{pmatrix} \mathfrak{p}_{00} & \cdots & \mathfrak{p}_{0m} \\ \vdots & & \vdots \\ \mathfrak{p}_{m0} & \cdots & \mathfrak{p}_{mm} \end{pmatrix}. \tag{1.15}$$

THEOREM 1.6. *Let $\mathfrak{f} = (f_0, \ldots, f_m)$ be a weakly perfect system. If the matrices P and $\mathfrak{P}$ of polynomials are defined as in (1.12) - (1.15), then we have*

$$P\,\mathfrak{P}^{\,t} = z^{(m+1)k} I_{m+1} \tag{1.16}$$

with I_{m+1} denoting the identity matrix with $m+1$ rows and columns.

Remark. Identity (1.16) is called a Mahler relation. It shows that at least in case of weakly perfect systems there exists a functional relationship between the Hermite-Padé polynomials of both types.

We shall see in the more detailed investigation of Hermite-Padé polynomials associated with the exponential function that despite of the functional relationship the properties of both types of polynomials can be rather different.

1.6. Hermite-Padé Polynomials to the Exponential Function. In the second part of the talk we are concerned with the asymptotic behavior of quadratic Hermite-Padé polynomials associated with the exponential function. In a general situation a system of exponential functions is of the form

$$\mathfrak{f}(z) = (f_0(z), \ldots, f_m(z)) := (1, e^z, \ldots, e^{mz}), \quad j = 1, \ldots, m. \tag{1.17}$$

From Definition 1.1 we know that for a multi-index $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ the corresponding Hermite-Padé polynomials $p_{j,n} \in \mathcal{P}_{n_j - 1}$, $j = 1, \ldots, m$, of type I are now defined by the relation

$$p_{0,n}(z) + p_{1,n}(z)e^z + \ldots + p_{m,n}(z)e^{mz} = O(z^{|n|-1}) \quad \text{as} \quad z \to 0, \tag{1.18}$$

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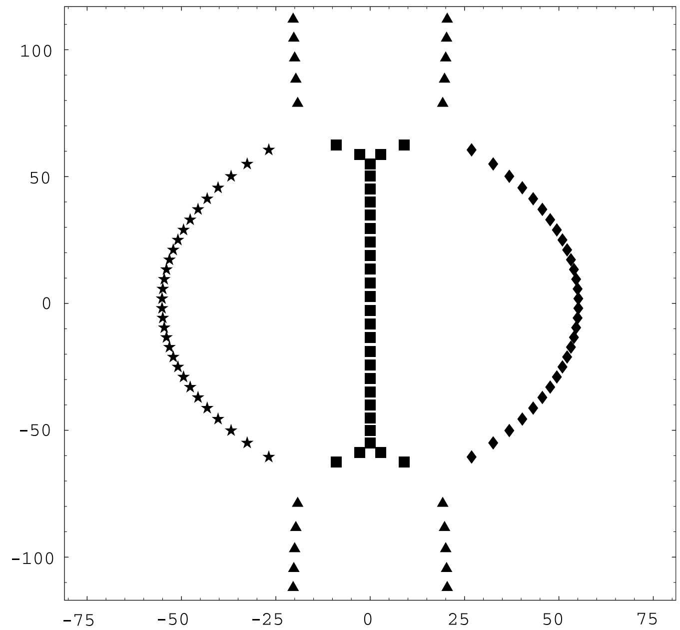

FIG. 1.1. *The zeros of the polynomials p_{30} (stars), q_{30} (boxes), r_{30} (diamonds), and some of the zeros of the error term e_{30} (triangles). (Notice that the axes have different scales.)*

and the Hermite-Padé polynomials $\mathfrak{p}_{j,n} \in \mathcal{P}_{|n|-n_j}$, $j = 1, \ldots, m$, of type II are defined by the m relations

$$\mathfrak{p}_{j,n}(z) - \mathfrak{p}_{0,n}(z)e^{jz} = O(z^{|n|+1}) \quad \text{as} \quad z \to 0, \quad j = 1, \ldots, m. \tag{1.19}$$

The polynomials of both types have been introduced in [10], and they have been used and intensively studied in number theory and approximation theory (cf. [14]-[16], [7], [24]).

1.7. Quadratic Hermite-Padé Polynomials. After moving from the general problem to the more special situation of a system of exponential functions, we continue to specialize, and restrict our interest now to quadratic Hermite-Padé polynomials, i.e., to the case $m = 2$, which is the simplest situation that does not coincide with the much studied and well understood case of Padé approximants to the exponential function. The (diagonal, quadratic) Hermite-Padé polynomials of type I associated with the exponential function e^z are denoted by $p_n, q_n, r_n \in \mathcal{P}_n$, and they are defined by the relation

$$e_n(z) := p_n(z) + q_n(z)e^z + r_n(z)e^{2z} = O(z^{3n+2}) \quad \text{as} \quad z \to 0. \tag{1.20}$$

The (diagonal, quadratic) Hermite-Padé polynomials of type II of degree $2n$ are denoted by $\mathfrak{p}_{2n}, \mathfrak{q}_{2n}, \mathfrak{r}_{2n} \in \mathcal{P}_{2n}$, and corresponding to Definition 1.2, they are defined by the two relations

$$\mathfrak{e}_{1,2n}(z) := \mathfrak{p}_{2n}(z)e^z - \mathfrak{q}_{2n}(z) = O(z^{3n+1}) \quad \text{as} \quad z \to 0, \tag{1.21}$$

$$\mathfrak{e}_{2,2n}(z) := \mathfrak{p}_{2n}(z)e^{2z} - \mathfrak{r}_{2n}(z) = O(z^{3n+1}) \quad \text{as} \quad z \to 0. \tag{1.22}$$

Note that in (1.20) the polynomials p_n, q_n, r_n are of degree n, and not of degree $n-1$, as assumed in Definition 1.1.

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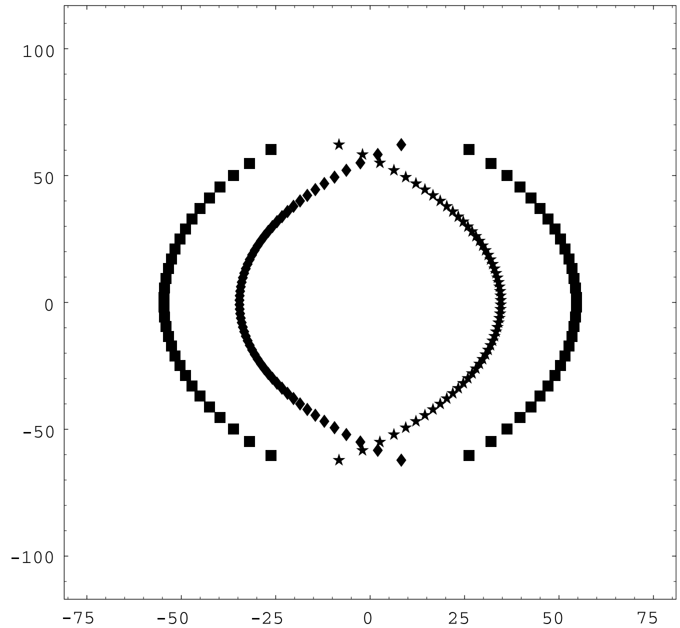

FIG. 1.2. *The zeros of the polynomials $\mathfrak{p}_{60}$ (stars), $\mathfrak{q}_{60}$ (boxes), $\mathfrak{r}_{60}$ (diamonds). (Notice that the axis have different scales.)*

The polynomials p_n, q_n, r_n are basic for the definition of quadratic approximants

$$\alpha_n(z) \ := \ \frac{1}{2r_n(z)}\left(-q_n(z) \pm \sqrt{q_n(z)^2 - p_n(z)r_n(z)}\right) \tag{1.23}$$

to e^z developed at $z = 0$, and the polynomials $\mathfrak{p}_{2n}$, $\mathfrak{q}_{2n}$, $\mathfrak{r}_{2n}$ lead to simultaneous rational approximants to the system (e^z, e^{2z}) defined by

$$\mathfrak{r}_{1,2n}(z) \ := \ \mathfrak{q}_{2n}/\mathfrak{p}_{2n}(z) \quad \text{and} \quad \mathfrak{r}_{2,2n}(z) \ := \ \mathfrak{r}_{2n}/\mathfrak{p}_{2n}(z). \tag{1.24}$$

These are the two types of approximants that have been introduced in the Definitions 1.4 and 1.3.

In the remainder of the talk the asymptotic behavior of the polynomials of both types will be investigated for $n \to \infty$. Detailed studies of the polynomials p_n, q_n, r_n can be found in [6], [8], and [9]. In [6] among other things a 4-term recurrence relation and very precise asymptotic estimates for the polynomials p_n, q_n, r_n and for the error term e_n have been derived. While in [6], like in (1.20), only the diagonal case has been studied, the investigations have been extended to the non-diagonal case in [8] and [9]. Interesting connections with special functions have been established in [9], and the paper contains results about the location of the zeros of the polynomials p_n, q_n, and r_n. Results achieved in [6], [8], and [9] have been extended to the general case (1.18) in [24].

In Figure 1.1 the zeros of the Hermite-Padé polynomials p_n, q_n, r_n, together with some of the zeros of the error term e_n in (1.20) are plotted for $n = 30$. In Figure 2.1 the zeros of the Hermite-Padé polynomials $\mathfrak{p}_{2n}$, $\mathfrak{q}_{2n}$, $\mathfrak{r}_{2n}$ of type II are plotted again for $n = 30$. The regularity displayed in these plots certainly suggests that there should exist analytic expressions that allow to describe the asymptotic behavior of the zeros, and also the asymptotic behavior of the polynomials p_n, q_n, r_n, $\mathfrak{p}_{2n}$, $\mathfrak{q}_{2n}$, $\mathfrak{r}_{2n}$ and the error term e_n themselves. Results in this direction are the main topic of the next two sections.

2. Asymptotics of Quadratic Hermite-Padé Polynomials. The zeros of the polynomials p_n, r_n, $\mathfrak{p}_{2n}$, $\mathfrak{q}_{2n}$, $\mathfrak{r}_{2n}$, and nearly all zeros of the polynomial q_n tend to infinity as $n \to \infty$. Because of this convergence to infinity, many specific aspects of the asymptotic behavior can not be detected in the complex plane. The asymptotic zero distributions and the asymptotics for the polynomials themselves become more informative if the independent variable z is rescaled in such a way that the zeros of the transformed polynomials have finite cluster points as $n \to \infty$. This concept has been used successfully by Szegö in [22] for the study of the asymptotic behavior of Taylor polynomials associated with the exponential function, and by Saff and Varga in [21] for the study of zeros and poles of Padé polynomials associated with the exponential function. In the same spirit as in these investigations we introduce as a new independent variable

$$w := \frac{z}{3n}, \quad n = 1, 2, \ldots, \tag{2.1}$$

for the study of the quadratic Hermite-Padé polynomials that will be presented in this talk. The (transformed) polynomials P_n, Q_n, R_n, $\mathfrak{P}_n$, $\mathfrak{Q}_n$, and $\mathfrak{R}_n$ are then defined by

$$P_n(w) := p_n(3nw), \quad Q_n(w) := q_n(3nw), \quad R_n(w) := r_n(3nw), \tag{2.2}$$

$$\mathfrak{P}_n(w) := \mathfrak{p}_n(3nw), \quad \mathfrak{Q}_n(w) := \mathfrak{q}_n(3nw), \quad \mathfrak{R}_n(w) := \mathfrak{r}_n(3nw). \tag{2.3}$$

These new polynomials satisfy the relations

$$E_n(w) := P_n(w)\left(e^{-3w}\right)^n + Q_n(w) + R_n(w)\left(e^{3w}\right)^n = O(z^{3n+2}), \tag{2.4}$$

$$\mathfrak{Q}_{2n}(w)(e^{-3w})^n - \mathfrak{P}_{2n}(w) = O(w^{3n+1}), \tag{2.5}$$

$$\mathfrak{Q}_{2n}(w)(e^{3w})^n - \mathfrak{R}_{2n}(w) = O(w^{3n+1}) \quad \text{as} \quad w \to 0. \tag{2.6}$$

These relations are equivalent to the relations (1.20) - (1.22) in the last section. The error term E_n in (1.20) is related to e_n by $E_n(w) = e_n(3nw)e^{-3nw}$. Note that in (1.20) not only the variable z has been substituted by $3nw$, but the relation has also been multiplied by e^{-w} in order to make the symmetry more evident, which is intrinsic to the problem. In a similar way the relations (1.21) and (1.22) have been transformed in order to get (2.5) and (2.6).

The polynomials P_n, Q_n, R_n, and $\mathfrak{P}_{2n}$, $\mathfrak{Q}_{2n}$, $\mathfrak{R}_{2n}$ are normalized by assuming that

$$P_n(w) = w^n + \ldots \in \mathcal{P}_n \quad \text{and} \quad \mathfrak{P}_{2n}(w) = w^{2n} + \ldots \in \mathcal{P}_{2n}. \tag{2.7}$$

Since the system $(e^{-w}, 1, e^w)$ is perfect (cf. [16]) the polynomials P_n, Q_n, R_n, and $\mathfrak{P}_{2n}$, $\mathfrak{Q}_{2n}$, $\mathfrak{R}_{2n}$ are uniquely determined by (2.4) - (2.7). It then follows from (2.4) that $P_n(w) = R_n(-w)$ and $Q_n(w) = Q_n(-w)$, and from (2.5) and (2.6) it follows that $\mathfrak{P}_n(w) = \mathfrak{R}_n(-w)$ and $\mathfrak{Q}_n(w) = \mathfrak{Q}_n(-w)$.

2.1. Asymptotic Distributions of Zeros. By $Z(p)$ we denote the (multi) set of zeros of a polynomial $p \in \mathcal{P}_n$ (multiplicities of zeros are represented by repetition), by ν_p the counting measure

$$\nu_p := \sum_{x \in Z(p)} \delta_x, \quad p \in \mathcal{P}_n, \tag{2.8}$$

and by $\stackrel{*}{\longrightarrow}$ the weak convergence of measures in $\overline{\mathbb{C}}$, i.e., $\mu_n \stackrel{*}{\longrightarrow} \mu$ means that $\int f d\mu_n \longrightarrow \int f d\mu$ holds for every real function f continuous on $\overline{\mathbb{C}}$, as $n \to \infty$.

The aim in the present investigation is to give an analytic interpretation of the regular configurations of zeros that can be observed in the Figures 1.1 and 1.2. Only parts of the result will be proved in the talk. These are mainly the statements connected with the definition of the asymptotic expressions. On the other hand the asymptotic relations themselves will not be proved. Thus, for instance, the existence of asymptotic distributions for the zeros in the Figures 1.1 and 1.2 is only conjectured.

CONJECTURE 2.1. *There exist six probability measures $\omega_P, \omega_Q, \omega_R, \omega_{\mathfrak{P}}, \omega_{\mathfrak{Q}}, \omega_{\mathfrak{R}}$ on $\overline{\mathbb{C}}$ such that the limits*

$$\frac{1}{n}\nu_{P_n} \stackrel{*}{\longrightarrow} \omega_P, \quad \frac{1}{n}\nu_{Q_n} \stackrel{*}{\longrightarrow} \omega_Q, \quad \frac{1}{n}\nu_{R_n} \stackrel{*}{\longrightarrow} \omega_R, \tag{2.9}$$

$$\frac{1}{2n}\nu_{\mathfrak{P}_n} \stackrel{*}{\longrightarrow} \omega_{\mathfrak{P}}, \quad \frac{1}{2n}\nu_{\mathfrak{Q}_n} \stackrel{*}{\longrightarrow} \omega_{\mathfrak{Q}}, \quad \frac{1}{2n}\nu_{\mathfrak{R}_n} \stackrel{*}{\longrightarrow} \omega_{\mathfrak{R}} \tag{2.10}$$

hold true as $n \to \infty$.

The supports of the measures ω_i, $i \in \{P, Q, R, \mathfrak{P}, \mathfrak{Q}, \mathfrak{R}\}$, are analytic arcs or the union of analytic arcs. The definition of theses arcs is one of the topics in the next subsection.

2.2. The Riemann Surface $\mathcal{R}$. We start by defining a Riemann surface $\mathcal{R}$ with three sheets and genus zero together with an algebraic function $\psi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$, which maps $\mathcal{R}$ bijectively on $\overline{\mathbb{C}}$. It will turn out that this Riemann surface is fundamental for all definitions relevant for the description of the asymptotics of the polynomials P_n, Q_n, R_n, $\mathfrak{P}_{2n}$, $\mathfrak{Q}_{2n}$, $\mathfrak{R}_{2n}$.

The Riemann surface $\mathcal{R}$ is introduced in order to make the function

$$v \mapsto w = w(v) := \frac{v^2 - 1/3}{v(v^2-1)}, \qquad v \in \overline{\mathbb{C}}, \tag{2.11}$$

bijective. Hence, $\mathcal{R}$ is the Riemann surface with canonical projection $\pi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ and the property that there exists a bijection $\psi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ such that $\pi \circ \psi^{-1}(v) = w(v)$ for all $v \in \overline{\mathbb{C}}$. The last requirement fully determines the surface $\mathcal{R}$ and the mapping ψ. The function ψ is algebraic of third degree. The surface $\mathcal{R}$ has three sheets and four simple branch points ζ_j, $j = 1, \ldots, 4$, over the four base points

$$w_j := \sqrt[4]{1/3}e^{i\varphi_j} \quad \text{with} \quad \varphi_j = \frac{5}{12}\pi, \frac{7}{12}\pi, \frac{17}{12}\pi, \frac{19}{12}\pi, \; j = 1, \ldots, 4. \tag{2.12}$$

Indeed, the derivative

$$w(v)' \;=\; -\frac{v^4 + 1/3}{v^2(v^2-1)^2} \tag{2.13}$$

has simple zeros at the four roots $v_j = \sqrt[4]{-1/3}$, $j = 1, \ldots, 4$, and it is easy to check that the four points in (2.12) are defined by $w_j = \pi \circ \psi^{-1}(v_j)$ for $j = 1, \ldots, 4$ if the v_j's are ordered appropriately. We note that for a given $\zeta \in \mathcal{R}$ the value $v = \psi(\zeta)$ can be calculated very efficiently by solving the cubic equation

$$\pi(\zeta)\, v\, (v^2 - 1) - v^2 + \frac{1}{3} \;=\; 0. \tag{2.14}$$

The value $v = \psi(\zeta)$ then is one of the three solutions of (2.14). The selection is determined by the sheet on which the point ζ lies.

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The three sheets of $\mathcal{R}$ will be denoted by B_{-1}, B_0, B_1. For a point $\zeta \in \mathcal{R}$ we write $\zeta^{(j)}$ if ζ lies on the sheet B_j. The canonical projection $\pi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is bijective on each sheet. By $\pi_j^{-1} : \overline{\mathbb{C}} \longrightarrow B_j$, $j = -1, 0, 1$, we denote the three branches of the inverse function π^{-1}. Points on $\mathcal{R}$ will generally be denoted by ζ, and by $\zeta^{(j)}$ if it is clear on which sheet B_j the point is lying. Base points will generally be denoted by w. For brevity we call the four points $\zeta_j \in \mathcal{R}$, $j = 1, \ldots, 4$, and also their four base points (2.12) branch points.

The information given so far leaves the three sheets rather arbitrary, but their definition will become more concrete as the analysis advances. In a first step we assume that the two sheets B_1 and B_0 are pasted together cross-wise in the usual way along a closed curve C_1 on $\mathcal{R}$, which is lying over an arc $\Gamma_1 \subseteq \mathbb{C}$ that connects the two branch points w_1 and w_4. It is assumed that Γ_1 intersects $\mathbb{R}$ between 0 and ∞. Analogously, the two sheets B_{-1} and B_0 are pasted together cross-wise along a curve C_{-1} lying over an arc $\Gamma_{-1} \subseteq \mathbb{C}$. The arc Γ_{-1} connects the two branch points w_2 with w_3 and intersects $\mathbb{R}$ between $-\infty$ and 0. Except for the assumptions about the intersections with $\mathbb{R}$ and the specific connections of the branch points, the arcs Γ_1 and Γ_{-1} are still fully arbitrary. (They will be determined in Lemma 2.4, below.)

It is not difficult to deduce from (2.11) that the numbering of the sheets can be done in such a way that

$$\psi(0^{(0)}) \; = \; \infty. \tag{2.15}$$

It then follows from (2.11) that $\psi(0^{(j)}) = j\sqrt{1/3}$ for $j = -1, 1$, and $\psi(\infty^{(j)}) = j$ for $j = -1, 0, 1$.

2.3. Definition of the Functions h_j. In the next step we shall show that there exists a function u such that the three branches of the function $h = \mathrm{Re}(u \circ \psi)$ have developments near infinity that model the asymptotic behavior of the three terms $\frac{1}{n}\log|P_n(w)\, e^{-3nw}|$, $\frac{1}{n}\log|Q_n(w)|$, $\frac{1}{n}\log|R_n(w)\, e^{3nw}|$ as $n \to \infty$ with P_n, Q_n, R_n defined by relation (2.2). In the next lemma it is shown that the function h and thereby also the function u is uniquely determined by properties that follow immediately from (2.4) and (2.7).

LEMMA 2.2. *Let h be a function harmonic in $\mathcal{R}\setminus\{\infty^{(-1)}, \infty^{(0)}, \infty^{(1)}, 0^{(0)}\}$ and assume that*

$$h(\zeta) = -3\,\mathrm{Re}\,\pi(\zeta) + \log|\pi(\zeta)| + \mathrm{O}\Big(\frac{1}{\pi(\zeta)}\Big) \quad \textit{as} \quad \zeta \to \infty^{(-1)}, \tag{2.16}$$

$$h(\zeta) = \log|\pi(\zeta)| + \mathrm{O}(1) \quad \textit{as} \quad \zeta \to \infty^{(0)}, \tag{2.17}$$

$$h(\zeta) = 3\,\mathrm{Re}(\pi(\zeta)) + \log|\pi(\zeta)| + \mathrm{O}(1) \quad \textit{as} \quad \zeta \to \infty^{(1)}, \tag{2.18}$$

$$h(\zeta) = 3\,\log|\pi(\zeta)| + \mathrm{O}(1) \quad \textit{as} \quad \zeta \to 0^{(0)}. \tag{2.19}$$

Then the function h is uniquely determined, and it is given by $h = \mathrm{Re}(u \circ \psi)$ with

$$u(v) := \frac{2\,v^2}{v^2 - 1} + \log\frac{2}{3\,v\,(v^2 - 1)}. \tag{2.20}$$

DEFINITION 2.3. *By*

$$\tilde{h}_j(w) = h \circ \pi_j^{-1}(w) \quad \textit{for} \quad j = -1, 0, 1, \; \textit{and } w \textit{ near } \infty, \tag{2.21}$$

$$\tilde{h}_\infty(w) = h \circ \pi_0^{-1}(w) \quad \textit{for} \quad w \textit{ near } 0 \tag{2.22}$$

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we define four harmonic function elements in neighborhoods of ∞ *and* 0*. In (2.21) and (2.22) the* π_j^{-1}*,* $j=-1,0,1$*, are the three branches of the inverse* π^{-1} *of the canonical projection* $\pi:\mathcal{R}\longrightarrow\overline{\mathbb{C}}$ *associated with the sheets* B_{-1}, B_0*, and* B_1*.*

Remarks. (1) Up to now the sheets B_{-1}, B_0, and B_1 are still rather arbitrary; however, the assumptions made about the arcs Γ_{-1} and Γ_1 guarantee that the functions π_j^{-1}, $j=-1,0,1$, are well defined in neighborhoods of ∞ and 0, and therefore the function elements (2.21) and (2.22) are well defined. Their global definition, however, depends on the specification of the sheets B_0, $j=-1,0,1$, which will be done in Lemma 2.4 below.

(2) The normalization (2.7) implies that $\frac{1}{n}\log|P_n(w)|=\log|w|+\mathrm{O}(1/w)$ as $w\to\infty$, and consequently we have $\frac{1}{n}\log|P_n(w)e^{-3nw}|=-3\,\mathrm{Re}(w)+\log|w|+\mathrm{O}(1/w)$ as $w\to\infty$, which corresponds to (2.16). In the same way we see that the two other terms in the middle part of relation (2.4) correspond to the expressions given in (2.17) and (2.18). In these last two relations we have an error term $\mathrm{O}(1)$ instead of $\mathrm{O}(1/w)$ since the normalization (2.7) is assumed only for the polynomial P_n. Since the term E_n in (2.4) has a zero of order at least $3n+2$ at 0, it follows that $\frac{1}{n}\log|E_n(w)|=3\,\log|w|+\mathrm{O}(1)$ as $w\to\infty$, and consequently also (2.19) follows directly from (2.5).

(3) Formula (2.20) allows one to derive as many terms in the developments of the function elements $\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1$, and $\tilde{h}_\infty$ as one wants.

(4) The function h is defined via the function u, which has been defined in (2.20) as a function in the $v-$plane. It is interesting and also important for the investigations below that the derivatives u' and $w'=(\pi\circ\psi^{-1})'$ have zeros at the same places in the $v-$plane. Indeed, we have

$$u'(v)=-\frac{3\,(v^4+1/3)}{v\,(v^2-1)^2}, \tag{2.23}$$

and a comparison with (2.13) shows that both functions have the same set of zeros.

No proofs will be given in the present section; all results that are not stated as conjectures will be proved in Section 3. This is also the case for Lemma 2.2.

In the next lemma arcs will be fixed with the help of assumptions about the global structure of the harmonic continuations of the four function elements $\tilde{h}_j$, $j=-1,0,1,\infty$. Among these arcs are the two arcs Γ_{-1} and Γ_1, which determine the sheets B_j, $j=-1,0,1$, of $\mathcal{R}$.

LEMMA 2.4. *(i) There exist uniquely two analytic Jordan arcs* Γ_{-1} *and* Γ_1 *such that the two function elements* $\tilde{h}_{-1}$ *and* $\tilde{h}_1$ *defined in (2.21) have harmonic continuations throughout the domains* $\mathbb{C}\setminus\Gamma_{-1}$ *and* $\mathbb{C}\setminus\Gamma_1$*, respectively, and the extended functions are continuous throughout* $\mathbb{C}$*. The arc* Γ_{-1} *connects the two branch points* w_2 *and* w_3 *in* $\{\mathrm{Re}(w)<0\}$*, and the arc* Γ_1 *connects the two branch points* w_1 *and* w_4 *in* $\{\mathrm{Re}(w)>0\}$*. The arc* Γ_1 *is the image of* Γ_{-1} *under reflection on the imaginary axis.*

(ii) There uniquely exists a continuum $K_0\subseteq\mathbb{C}$ *such that the function element* $\tilde{h}_0$ *defined in (2.21) has an harmonic continuation throughout the domain* $\mathbb{C}\setminus K_0$*, the extended function is continuous throughout* $\mathbb{C}$*, and the continuum* K_0 *has no subarcs in common with* Γ_{-1} *or* Γ_1*. The continuum* K_0 *is the union of five analytic Jordan arcs* $\Gamma_{00},\ldots,\Gamma_{04}$*, and it connects all four points* $w_1,\ldots,w_4$*. The subarc* Γ_{00} *is the interval* $[-i\,y_1, i\,y_1]$ *with* y_1 *a positive number. The subarcs* Γ_{01} *and* Γ_{02} *connect the two branch points* w_1 *and* w_2 *with* $i\,y_1$*, the subarcs* Γ_{03} *and* Γ_{04} *connect the branch points* w_3 *and* w_4 *with* $-i\,y_1$*. For* y_1 *we have the numerical value*

$$y_1 \doteq 0.621391 \tag{2.24}$$

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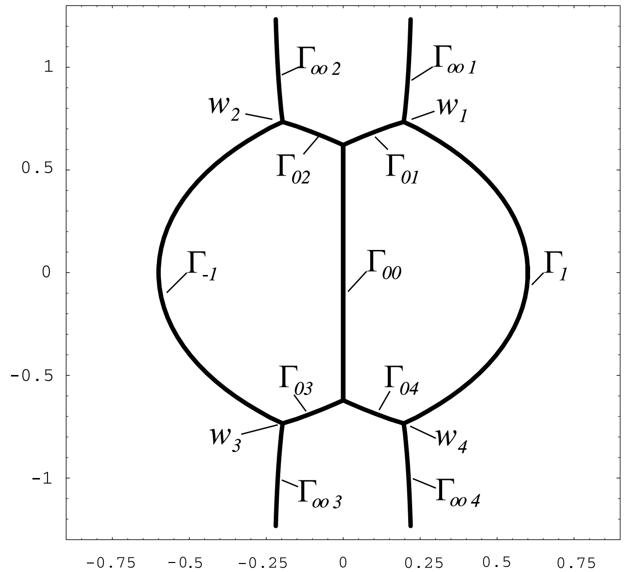

FIG. 2.1. *The arcs* Γ_{-1}, Γ_1, *the set* $K_0 = \Gamma_{00} \cup \ldots \cup \Gamma_{04}$, *and parts of the set* $K_\infty = \Gamma_{\infty 1} \cup \ldots \cup \Gamma_{\infty 4}$.

(iii) There uniquely exists a continuum $K_\infty \subseteq \overline{\mathbb{C}}$ *such that the function element* $\tilde{h}_\infty$ *defined in (2.22) has a harmonic continuation throughout the domain* $\mathbb{C} \setminus (\{0\} \cup K_\infty)$*, the extended function is continuous throughout* $\mathbb{C} \setminus \{0\}$*, and the continuum* K_∞ *has no subarcs in common with* Γ_{-1}, Γ_1, *or* K_0. *The continuum* K_∞ *is the union of four analytic Jordan arcs* $\Gamma_{\infty 1}, \ldots, \Gamma_{\infty 4} \subseteq \overline{\mathbb{C}}$, *each* $\Gamma_{\infty j}$, $j = 1, \ldots, 4$, *connects the branch point* w_j *with* ∞. *The* $\Gamma_{\infty 1}, \ldots, \Gamma_{\infty 4}$ *are disjoint in* $\mathbb{C}$.

DEFINITION 2.5. *By* h_j, $j = -1, 0, 1, \infty$, *we denote the harmonic continuations of the function elements* $\tilde{h}_j$ *into the domains* $\overline{\mathbb{C}} \setminus \Gamma_{-1}$, $\overline{\mathbb{C}} \setminus K_0$, $\overline{\mathbb{C}} \setminus \Gamma_1$, *and* $\overline{\mathbb{C}} \setminus K_\infty$, *respectively. Because of the continuity assumption, these continuations extend to the whole* $\overline{\mathbb{C}}$.

Remarks. (1) With the determination of the two Jordan arcs Γ_1 and Γ_{-1} in part (i) of the Lemma 2.4 the shape of the three sheets B_{-1}, B_0, B_1 of the surface $\mathcal{R}$ is finally fixed. The definition of the sheets is unique up to the attribution of the boundary curves C_{-1} and C_1 to each of the neighboring sheets. This can always be done in a satisfactory way.

(2) The arcs Γ_1, Γ_{-1}, the set K_0, and parts of the set K_∞ are shown in Figure 2.1.

(3) Below, in Theorem 2.16, tools will be introduced which allow to calculate all arcs mentioned in Lemma 2.4 in a very efficient way. The existence of these tools allows us to say that the arcs Γ_1, Γ_{-1}, and the sets K_0, K_∞ are defined in a constructive fashion.

(4) The surface $\mathcal{R}$ has three sheets, but in Lemma 2.4 we have considered four branches h_j, $j = -1, 0, 1, \infty$, of the function h. Consequently, everywhere in $\mathbb{C} \setminus (\Gamma_{-1} \cup K_0 \cup \Gamma_1 \cup K_\infty)$ two of the four branches have to be identical. The harmonicity implies that the identical pair has to be the same in each component of the set $\mathbb{C} \setminus (\Gamma_{-1} \cup K_0 \cup \Gamma_1 \cup K_\infty)$. The pairing can only change if w crosses one of the subarcs of $\Gamma_{-1} \cup K_0 \cup \Gamma_1 \cup K_\infty$.

2.4. Definition of the Measures ν_j. For the functions h_j, $j = -1, 0, 1$, of Definition 2.5 there exist representations which involve logarithmic potentials, and these potentials de-

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termine three probability measures that are the asymptotic distributions of the zeros of the polynomials P_n, Q_n, R_n.

LEMMA 2.6. *There exist three probability measures ν_1, ν_0, ν_{-1} such that*

$$h_{-1}(w) = -3\,\mathrm{Re}(w) - \int \log \frac{1}{|w-x|} d\nu_{-1}(x), \tag{2.25}$$

$$h_0(w) = \log(2) - \int \log \frac{1}{|w-x|} d\nu_0(x), \tag{2.26}$$

$$h_1(w) = 3\,\mathrm{Re}(w) - \int \log \frac{1}{|w-x|} d\nu_1(x). \tag{2.27}$$

We have supp$(\nu_j) = \Gamma_j$ for $j = -1, 1$, and supp$(\nu_0) = K_0$. The measure ν_{-1} is the image of ν_1 under reflection on the imaginary axis.

Remarks. (1) The measures ν_{-1}, ν_0, ν_1 are absolutely continuous with respect to linear Lebesgue measure on supp(ν_j) for $j = -1, 0, 1$, respectively. Below, in Theorem 2.17, tools will be presented that allow an efficient calculation of the density functions of these three measures.

(2) The density functions are real-analytic with respect to arc length inside of the arcs that form the supports. Near the branch points $w_1, \dots, w_4$ the density functions are of the form $const * \sqrt{dist(w, w_j)} + \mathrm{O}(|w - w_j|)$ for $w \in$ supp(ν_j) and $w \to w_j$, $j = 1, \dots, 4$.

2.5. Asymptotics I. The definitions of the last two subsections allow to formulate the first group of asymptotic results, which are a core piece of the present talk. These statements are still conjectures since parts of the proofs still have not been worked out.

CONJECTURE 2.7. *Let the functions h_j, $j = -1, 0, 1, \infty$, the arcs Γ_{-1}, Γ_1, and the continua K_0, K_∞, be defined as in the Lemma 2.4 and Definition 2.5. Then we have*

$$\lim_{n\to\infty} \frac{1}{n} \log |P_n(w)| = h_{-1}(w) + 3\,\mathrm{Re}(w) \text{ locally uniformly for } w \in \mathbb{C} \setminus \Gamma_{-1}, \tag{2.28}$$

$$\lim_{n\to\infty} \frac{1}{n} \log |Q_n(w)| = h_0(w) \text{ locally uniformly for } w \in \mathbb{C} \setminus K_0, \tag{2.29}$$

$$\lim_{n\to\infty} \frac{1}{n} \log |R_n(w)| = h_1(w) - 3\,\mathrm{Re}(w) \text{ locally uniformly for } w \in \mathbb{C} \setminus \Gamma_1, \tag{2.30}$$

$$\lim_{n\to\infty} \frac{1}{n} \log |E_n(w)| = h_\infty(w) \text{ locally uniformly for } w \in \mathbb{C} \setminus (\{0\} \cup K_\infty). \tag{2.31}$$

CONJECTURE 2.8. *The three measures $\omega_P, \omega_Q, \omega_R$ in Conjecture 2.1 are given by*

$$\omega_P = \nu_{-1}, \qquad \omega_Q = \nu_0, \qquad \omega_R = \nu_1, \tag{2.32}$$

where ν_j, $j = -1, 0, 1$, are the probability measures introduced in Lemma 2.6.

In Figure 2.2 the zeros from Figure 1.1 are plotted together with the arcs introduced in Lemma 2.4. The zeros in Figure 1.1 have been calculated in the $z-$variable. In order to make them comparable to the scales of Figure 2.1, Figure 1.1 has been transformed by the function (2.1) with $n = 30$. The arcs Γ_{-1}, Γ_1, and the set K_0 are the supports of the measures ν_j, $j = -1, 0, 1$. As Figure 2.2 shows there exists a good accordance between the zeros and the supports of their asymptotic distributions already for $n = 30$. In Conjecture 2.8 only weak convergence has been considered; in a forthcoming paper also a strong version of the

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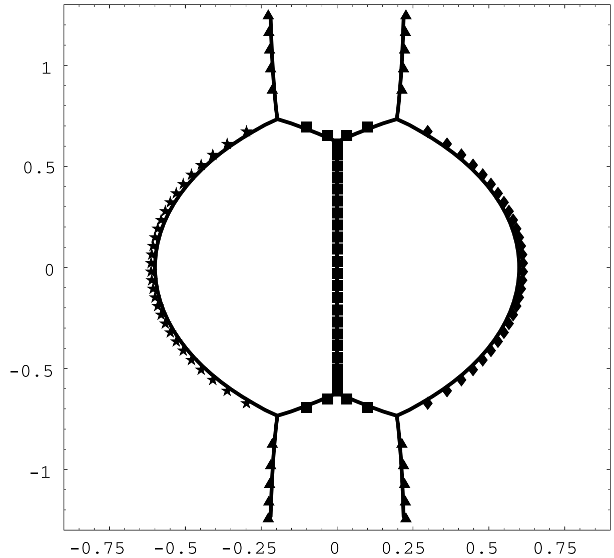

FIG. 2.2. *An overlay of Figure 1.1 with Figure 2.1 after a shrinking of the scales of Figure 1.1 in accordance with (2.1).*

asymptotic zero distributions will be proved. These strong asymptotic relations are precise enough to fix approximate positions of individual zeros.

Also for the zeros of the error term E_n an asymptotic distribution can be found with a method that is similar to that used to define the three probability measures ν_{-1}, ν_0, ν_1. However, this asymptotic distribution has no compact support and its mass is infinite. We shall not address the problem in the present talk.

2.6. Definition of the Functions g_j and the Measures ψ_j. Our next aim is to present asymptotic relations for the Hermite-Padé polynomials $\mathfrak{P}_{2n}$, $\mathfrak{Q}_{2n}$, $\mathfrak{R}_{2n}$ of type II. The approach will be analogous to that applied in the last two subsections for the asymptotic relation of the polynomials P_n, Q_n, R_n of type I. In a first step we define functions g_j and measures ψ_j, $j = -1, 0, 1$, which will be the building blocks of the asymptotic relations.

LEMMA 2.9. *There exist uniquely two analytic Jordan arcs $\Gamma_{-1,2}$, Γ_{12}, and a set K_1 such that*

(i) the three function elements $\tilde{h}_{-1}$, $\tilde{h}_1$, and $\tilde{h}_0$ defined in (2.21) have harmonic continuations g_{-1}, g_1, and g_0 throughout the domains $\mathbb{C} \setminus (\Gamma_{-1,2} \cup \{0\})$, $\mathbb{C} \setminus (\Gamma_{12} \cup \{0\})$, and $\mathbb{C} \setminus (K_1 \cup \{0\})$, respectively,

(ii) at the origin we have

$$g_j(w) = 3 \log |w| + O(1) \quad \textit{as} \quad w \to 0 \quad \textit{for} \quad j = -1, 0, 1, \ \textit{and} \tag{2.33}$$

(iii) the functions g_{-1}, g_0, g_1 extend continuously throughout $\mathbb{C} \setminus \{0\}$.

The function elements $\tilde{h}_{-1}$, $\tilde{h}_1$, and $\tilde{h}_0$ represent the three branches at infinity of the function h defined in Definition 2.3. Comparing Lemma 2.4 with Lemma 2.9 we see that near infinity three functions h_{-1}, h_0, h_1 and the newly introduced functions g_{-1}, g_0, g_1 are identical. However, globally these functions are different. The boundaries $\Gamma_{-1,2}$, K_1, Γ_{12} of the domains of definition of the functions g_{-1}, g_0, g_1 are defined by a similar principle as that applied in Lemma 2.4 for Γ_{-1}, Γ_1 and K_0. Therefore, it is not surprising that there exists a connection between the new arcs $\Gamma_{-1,2}$, Γ_{12} and set K_1 on one hand and the arcs Γ_{-1}, Γ_1 and set K_0 defined in Lemma 2.4 on the other hand. The connections are stated in the next lemma.

LEMMA 2.10. *(i) We have* $K_1 = \Gamma_{-1} \cup \Gamma_1$.

(ii) The arc $\Gamma_{-1,2}$ *connects the branch point* w_2 *with* w_3*, and the arc* Γ_{12} *the branch point* w_1 *with* w_4*. The subarcs* Γ_{02} *and* Γ_{03} *of* K_0 *introduced in Lemma 2.4 are subarcs of the arc* $\Gamma_{-1,2}$*, and the subarcs* Γ_{01} *and* Γ_{04} *of* K_0 *are subarcs of the arc* Γ_{12}.

(iii) The arc $\Gamma_{-1,2}$ *intersects* $\mathbb{R}$ *in the interval* $(0, \infty)$*, and the arc* Γ_{12} *intersects* $\mathbb{R}$ *in the interval* $(-\infty, 0)$.

Remark. The arcs $\Gamma_{-1,2}$, Γ_{12} and the set K_1 are plotted in Figure 2.3.

For the functions $g_j, j = -1, 0, 1$, there exist representations involving logarithmic potentials. With the help of these potentials we introduce the probability measures $\psi_j, j = -1, 0, 1$.

LEMMA 2.11. *There exist three probability measures* $\psi_{-1}, \psi_0, \psi_1$ *such that*

$$g_{-1}(w) = -3\,\mathrm{Re}(w) + 3\,\log|w| + 2\int \log\frac{1}{|w-x|}d\psi_{-1}(x), \tag{2.34}$$

$$g_0(w) = \log(2) + 3\,\log|w| + 2\int \log\frac{1}{|w-x|}d\psi_0(x), \tag{2.35}$$

$$g_1(w) = 3\,\mathrm{Re}(w) + 3\,\log|w| + 2\int \log\frac{1}{|w-x|}d\psi_1(x). \tag{2.36}$$

We have supp$(\psi_j) = \Gamma_{j,2}$ *for* $j = -1, 1$*, and supp*$(\psi_0) = K_1$*. The measure* ψ_{-1} *is the image of* ψ_1 *under reflection on the imaginary axis. The measure* ψ_0 *is symmetric with respect to the imaginary axis, and we have* $\psi_0 = (\nu_{-1} + \nu_1)/2$.

Remark. Like the measures ν_{-1}, ν_0, ν_1, the measures $\psi_{-1}, \psi_0, \psi_1$ are also absolutely continuous with respect to linear Lebesgue measure on the supports supp(ψ_j), $j = -1, 0, 1$. Below, in Theorem 2.17, tools will be presented that allow an efficient calculation of the density functions of the three measures.

2.7. Asymptotics II. With the definitions of the last subsection we are prepared to formulate the asymptotic relations for the Hermite-Padé polynomials $\mathfrak{P}_{2n}$, $\mathfrak{Q}_{2n}$, $\mathfrak{R}_{2n}$ of type II.

CONJECTURE 2.12. *Let the functions* g_j*,* $j = -1, 0, 1$*, the arcs* $\Gamma_{-1,2}$*,* Γ_{12}*, and the set* K_1*, be defined as in the Lemma 2.9. Then locally uniformly we have*

$$\lim_{n\to\infty}\frac{1}{n}\log|\mathfrak{P}_{2n}(w)| = -g_{-1}(w) + 3\log|w| + 3\,\mathrm{Re}(w) \quad \text{for } w \in \mathbb{C}\setminus(\Gamma_{-1,2}\cup\{0\}), \tag{2.37}$$

$$\lim_{n\to\infty}\frac{1}{n}\log|\mathfrak{Q}_{2n}(w)| = -g_0(w) + 3\log|w| \quad \text{for } w \in \mathbb{C}\setminus(K_1\cup\{0\}), \tag{2.38}$$

$$\lim_{n\to\infty}\frac{1}{n}\log|\mathfrak{R}_{2n}(w)| = -g_1(w) + 3\log|w| - 3\,\mathrm{Re}(w) \text{ for } w \in \mathbb{C}\setminus(\Gamma_{12}\cup\{0\}). \tag{2.39}$$

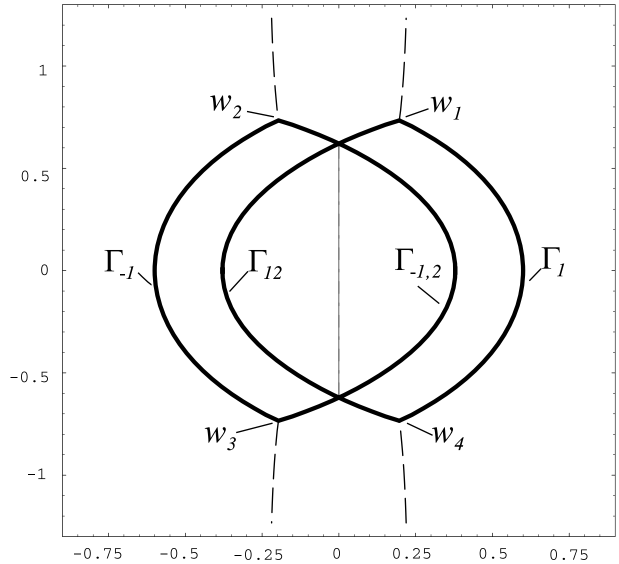

FIG. 2.3. *The arcs $\Gamma_{-1,2}$, Γ_{12} and the set $K_1 = \Gamma_{-1} \cup \Gamma_1$. The arcs of Figure 2.1 are included into the figure and represented by dashed lines (— — —).*

CONJECTURE 2.13. *The three measures $\omega_{\mathfrak{P}}, \omega_{\mathfrak{Q}}, \omega_{\mathfrak{R}}$ in Conjecture 2.1 are given by*

$$\omega_{\mathfrak{P}} = \psi_{-1}, \qquad \omega_{\mathfrak{Q}} = \psi_0, \qquad \omega_{\mathfrak{R}} = \psi_1, \tag{2.40}$$

where ψ_j, $j = -1, 0, 1$, are the probability measures introduced in Lemma 2.11.

In Figure 2.4 the zeros of the Hermite-Padé polynomials $\mathfrak{P}_{60}$, $\mathfrak{Q}_{60}$, $\mathfrak{R}_{60}$ of type II are plotted together with the arcs introduced in Lemma 2.9 and plotted in Figure 2.3. Note that Figure 1.1 has been rescaled by function (2.1) with $n = 30$. As in Figure 2.2, we observe a good approximation of the support $\mathrm{supp}(\psi_j)$, $j = -1, 0, 1$, by the zeros of the Hermite-Padé polynomials $\mathfrak{P}_{60}$, $\mathfrak{Q}_{60}$, $\mathfrak{R}_{60}$, respectively.

2.8. A Numerical Method. In the last subsection we presented a numerical method which allows to calculate the functions h_j, $j = -1, 0, 1, \infty$, and g_j, $j = -1, 0, 1$, in the asymptotic relations of the Conjectures 2.7 and 2.12, the arcs $\Gamma_{-1}, \ldots, \Gamma_{12}$ introduced in the Lemmas 2.4 and 2.9, and the density functions of the measures $\nu_{-1}, \nu_0, \nu_1, \psi_{-1}, \psi_0, \psi_1$ introduced in the Lemmas 2.6 and 2.11.

The function h defined on the Riemann surface $\mathcal{R}$ and introduced in Lemma 2.2 is basic for the definition of the functions h_j and g_j. Let f denote the function

$$f(v) := \mathrm{Re}\, \frac{2\, v^2}{v^2 - 1} + \log \frac{2}{3} - \log \left| v(v^2 - 1) \right|, \tag{2.41}$$

which is the real part of function (2.20) in Lemma 2.2. From Lemma 2.2 and the definition of the functions h_j and g_j in the Lemmas 2.4 and 2.9 it follows that for any given $w \in \mathbb{C}$ the values $h_j(w)$ and $g_j(w)$ are equal to $f(v_j)$ with $v_j = \psi \circ \pi_{l_j}^{-1}(w)$ if the branch $\pi_{l_j}^{-1}$ of the inverse projection π^{-1} is chosen appropriately. The calculation of $\psi \circ \pi_l^{-1}(w)$ can be done

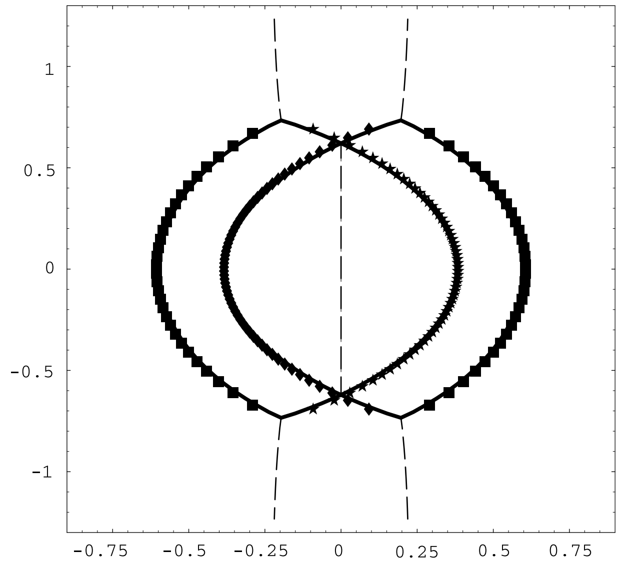

FIG. 2.4. *An overlay of Figure 1.2 with Figure 2.3 after a shrinking of the scales of Figure 1.2 in accordance with (2.1).*

very efficiently, as we have seen in Subsection 2.2. The value of $\psi \circ \pi_l^{-1}(w)$ is one of the three solutions v_l, $l = -1, 0, 1$, of the equation

$$w\, v\, (v^2 - 1) - v^2 + \frac{1}{3} = 0. \tag{2.42}$$

However, the selection of the right solution is a problem that is equivalent to choosing the appropriate branch of π^{-1} in $\psi \circ \pi_{l_j}^{-1}(w)$. In the next theorem we present a strategy for making these selections by continuation along chains of points in the domains of definition of the functions h_j and g_j. Let D_j, $j = -1, 0, 1, \infty$, denote the domain of definition of the function h_j, i.e., by Lemma 2.4 we have $D_j := \overline{\mathbb{C}} \setminus \Gamma_j$ for $j = -1, 1$, and $D_j := \overline{\mathbb{C}} \setminus K_j$ for $j = 0, \infty$. Let G_j, $j = -1, 0, 1$, further denote the domain of definition of the function g_j, i.e., by Lemma 2.9 we have $G_j := \overline{\mathbb{C}} \setminus (\Gamma_{j2} \cup \{0\})$ for $j = -1, 1$, and $G_0 := \overline{\mathbb{C}} \setminus (K_1 \cup \{0\})$.

THEOREM 2.14. *(i) Let $w \in \mathbb{C}$ lie close to ∞. Then we can choose the index l of the three solutions v_l of equation (2.42) in such a way that v_l lies close to l for each $l = -1, 0, 1$, and it follows that*

$$h_j(w) = g_j(w) = f(v_j) \quad \textit{for} \quad j = -1, 0, 1. \tag{2.43}$$

(ii) Let $w \in \mathbb{C}$ lie close to 0. Then there exists exactly one solution v_∞ of equation (2.42) that lies close to ∞ (the other two solutions lie close to $\pm\sqrt{1/3}$). With this choice of v_∞ it follows that

$$h_\infty(w) = f(v_\infty). \tag{2.44}$$

(iii) For an arbitrary $w \in D_j$, $j = -1, 0, 1$, or $w \in G_j$, $j = -1, 0, 1$, the values $h_j(w)$ or $g_j(w)$, respectively, can be calculated by continuation of these functions through D_j or G_j starting from a neighborhood of infinity. If we assume that for a point $w = w_0 \in D_j$

the solution $v_j = v_j(w_0)$ *of equation (2.42) has been chosen in such a way that* $h_j(w_0) = f(v_j(w_0))$*, and if* $w_1 \in D_j$ *lies close to* w_0*, then one has to choose, from the three roots of equation (2.42) with* $w = w_1$*, the solution* $v_j = v_j(w_1)$ *that lies closest to* $v_j = v_j(w_0)$*. For this new* v_j *one has*

$$h_j(w_1) = f(v_j(w_1)). \tag{2.45}$$

In an analogous way one can proceed for the functions g_j *and* $w \in G_j$*,* $j = -1, 0, 1$*.*

(iv) The function h_∞ *is calculated by continuation in* D_∞ *using the same strategy as that described in (iii), only that now one has to start from a neighborhood of* $w = 0$*.*

Next we address the problem of calculating the arcs introduced in Lemmas 2.4 and 2.9.

DEFINITION 2.15. *In order to simplify notation, we define* $\Gamma_{11} := \Gamma_1$*,* $\Gamma_{-1,1} := \Gamma_{-1}$*,* $\Gamma_{13} := \Gamma_{\infty 1} \cup \Gamma_{\infty 4}$*,* $\Gamma_{-1,3} := \Gamma_{\infty 2} \cup \Gamma_{\infty 3}$*. (The arcs* $\Gamma_{-1,2}$ *and* Γ_{12} *have already been defined in Lemma 2.9.)*

It is immediate that all arcs introduced in the Lemmas 2.4 and 2.9 are subarcs of the system Γ_{ji}, $j = -1, 1$, $i = 1, 2, 3$, together with $\Gamma_{00} = [-iy_1, iy_1]$. A numerical value for y_1 has been given in (2.24).

THEOREM 2.16. *(i) The three arcs* $\Gamma_{11}, \Gamma_{12}, \Gamma_{13}$ *are analytic, they are disjoint in* $\overline{\mathbb{C}} \setminus \{w_1, w_4\}$*, and each one of them connects* w_1 *with* w_4 *in* $\overline{\mathbb{C}}$*. All three arcs are symmetric with respect to* $\mathbb{R}$*, and we have* $\infty \in \Gamma_{13}$*. At* w_1 *the arcs* $\Gamma_{11}, \Gamma_{12}, \Gamma_{13}$ *have tangential directions*

$$\varphi_1 = 65\,\pi/36, \qquad \varphi_2 = 41\,\pi/36, \qquad \varphi_3 = 17\,\pi/36, \tag{2.46}$$

respectively. The tangential directions at w_4 *follow from (2.46) and the symmetry with respect to* $\mathbb{R}$*.*

(ii) For $w = w_1$ *and* $w = w_4$ *equation (2.42) has two identical solutions. For each* $w \in \Gamma_{1j}$*,* $j = 1, 2, 3$*, a pair can be selected from the three solutions of equation (2.42) in such a way that the two elements of this pair are a continuation along* Γ_{1j} *of the two elements of the identical pair at* w_1 *or* w_4*. The two solutions selected in that way are denoted by* $v_{j+} = v_{j+}(w)$ *and* $v_{j-} = v_{j-}(w)$ *for* $w \in \Gamma_{1j}$*. Let* $t_w \in \partial\mathbb{D}$ *denote the tangent on the arc* Γ_{1j} *at the point* $w \in \Gamma_{1j}$*. Then we have*

$$t_w = \pm i \frac{\overline{v_{j+}(w) - v_{j-}(w)}}{|v_{j+}(w) - v_{j-}(w)|} \qquad \textit{for} \qquad w \in \Gamma_{1j},\ j = 1, 2, 3. \tag{2.47}$$

(iii) The three arcs $\Gamma_{-1,j}$*,* $j = 1, 2, 3$*, are the images of the arcs* Γ_{1j}*,* $j = 1, 2, 3$*, under reflection on the imaginary axis.*

(iv) For the subarcs of K_0 *and* K_∞ *introduced in Lemma 2.4 we have*

$$\Gamma_{01} = \Gamma_{12} \cap \{\mathrm{Re}(w) > 0, \mathrm{Im}(w) > 0\}\,, \tag{2.48}$$
$$\Gamma_{02} = \Gamma_{-1,2} \cap \{\mathrm{Re}(w) < 0, \mathrm{Im}(w) > 0\}\,, \tag{2.49}$$
$$\Gamma_{03} = \Gamma_{-1,2} \cap \{\mathrm{Re}(w) < 0, \mathrm{Im}(w) < 0\}\,, \tag{2.50}$$
$$\Gamma_{04} = \Gamma_{12} \cap \{\mathrm{Re}(w) > 0, \mathrm{Im}(w) < 0\}\,, \tag{2.51}$$
$$\Gamma_{\infty 1} = \Gamma_{13} \cap \{\mathrm{Im}(w) > 0\}\,, \quad \Gamma_{\infty 4} = \Gamma_{13} \cap \{\mathrm{Im}(w) < 0\}\,, \tag{2.52}$$
$$\Gamma_{\infty 2} = \Gamma_{-1,3} \cap \{\mathrm{Im}(w) > 0\}\,, \quad \Gamma_{\infty 3} = \Gamma_{-1,3} \cap \{\mathrm{Im}(w) < 0\}\,. \tag{2.53}$$

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Remark. The initial directions of the arcs Γ_{1j}, $j = 1, 2, 3$, at one of the branch points w_1 or w_4 given in part (i) of the theorem together with the formula (2.47) for tangent, form the basis for an efficient calculation of the Jordan arcs Γ_{1j}, $j = 1, 2, 3$. By symmetry this also allows to calculate the Jordan arcs $\Gamma_{-1,j}$, $j = 1, 2, 3$. The information given in part (iv) of the theorem allow to break down the arcs Γ_{ij} into those pieces, which are needed for the construction of the sets K_0, K_∞. The arcs shown in the Figures 2.1 - 2.4 are calculated in this way.

The last theorem in the present section contains a numerical method for calculating the density functions of the measures $\nu_{-1}, \ldots, \psi_1$. In a first step we introduce five new measures μ_{ij}, $i = -1, 1$, $j = 1, 2$, μ_{00} on the arcs Γ_{ij}, $\Gamma_{00} = [-iy_1, iy_1]$, respectively.

THEOREM 2.17. *(i) Let for $w \in \Gamma_{1j}$, $j = 1, 2$, the two solutions $v_{j+} = v_{j+}(w)$ and $v_{j-} = v_{j-}(w)$ of equation (2.42) be selected as described in part (ii) of Theorem 2.16. We define measures μ_{1j}, $j = 1, 2$, on Γ_{11} and Γ_{12} by*

$$d\mu_{1j}(w) := \frac{3}{2\pi} \left| v_{j+}(w) - v_{j-}(w) \right| ds_w, \qquad w \in \Gamma_{1j},\ j = 1, 2, \tag{2.54}$$

where ds_w denotes the line element on the arcs Γ_{1j}, $j = 1, 2$.

(ii) Let the two measures $\mu_{-1,j}$, $j = 1, 2$, be the images of the measures μ_{1j}, $j = 1, 2$, under the reflection on the imaginary axis.

(iii) For $w \in \Gamma_{00} = [-iy_1, iy_1]$ let $v_{0+} = v_{0+}(w)$ and $v_{0-} = v_{0-}(w)$ denote the two solutions of equation (2.42) that are symmetric with respect to the imaginary axis. The measure μ_{00} is defined on Γ_{00} by

$$d\mu_{00}(w) := \frac{3}{2\pi} \left| v_{0+}(w) - v_{0-}(w) \right| ds_w, \qquad w \in \Gamma_{00}. \tag{2.55}$$

With the definitions of part (i), (ii), and (iii) we have

$$\nu_j = \mu_{j1}, \qquad j = -1, 1, \tag{2.56}$$

$$\nu_0 = \mu_{00} + \mu_{-1,2}\big|_{\{\mathrm{Re}(w)<0\}} + \mu_{12}\big|_{\{\mathrm{Re}(w)>0\}}, \tag{2.57}$$

$$\psi_j = \frac{1}{2}\,\mu_{j2}, \qquad j = -1, 1, \tag{2.58}$$

$$\psi_0 = \frac{1}{2}\left(\mu_{-1,1} + \mu_{11}\right). \tag{2.59}$$

3. Proofs.

The lemmas 2.2, 2.4, 2.6, 2.9, 2.10, 2.11, and the Theorems 2.14, 2.16, and 2.17 are proved in the present section.

3.1. Proof of Lemma 2.2.

Let the function h be defined as $\mathrm{Re}(u \circ \psi)$ on $\mathcal{R}$ with u given by (2.20), then this function has the developments (2.16-2.19) at the points $\infty^{(-1)}$, $\infty^{(0)}$, $\infty^{(1)}$, and $0^{(0)}$. This can be verified by straightforward calculations using (2.11) for the definition of $\pi \circ \psi^{-1}$.

In order to prove uniqueness, we assume that g is another function that is harmonic in $\mathcal{R} \setminus \{\infty^{(-1)}, \infty^{(0)}, \infty^{(1)}, 0^{(0)}\}$ and satisfies the assumptions made in (2.16-2.19) for the function h. Then $h - g$ is harmonic throughout the compact surface $\mathcal{R}$, and consequently $h - g$ is constant. From (2.16) we know that $(h - g)(\infty^{(-1)}) = 0$, which proves uniqueness. $\square$

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3.2. Auxiliary Lemmas. Next we state and prove two auxiliary lemmas.

LEMMA 3.1. *We have*

$$(u \circ \psi \circ \pi_j^{-1})'(w) = 3\,(\psi \circ \pi_j^{-1})(w) = 3\,\psi(\zeta^{(j)}) \quad \text{for} \quad w \in \overline{\mathbb{C}},\ j = -1, 0, 1, \tag{3.1}$$

with u defined in (2.20), $w = \pi(\zeta^{(j)}),\ \zeta^{(j)} \in \mathcal{R}$.

Proof. From the chain rule it follows that

$$(u \circ \psi \circ \pi_j^{-1})'(w) = u'((\psi \circ \pi_j^{-1})(w))(\psi \circ \pi_j^{-1})'(w) = \frac{u'(v)}{(\pi \circ \psi^{-1})'(v)} \tag{3.2}$$

with $v = \psi(\zeta^{(j)})$. Using the expressions (2.13) and (2.23), this yields $(u \circ \psi \circ \pi_j^{-1})'(w) = 3\,v$, which proves (3.1). $\square$

LEMMA 3.2. *Define the set N_1 as*

$$N_1 := \left\{ w \in \overline{\mathbb{C}} \;\middle|\; h \circ \pi_1^{-1}(w) = h \circ \pi_0^{-1}(w) \right\}, \tag{3.3}$$

where the branches π_j^{-1}, $j = -1, 0, 1$, of π^{-1} are determined by the choice of the sheets B_{-1}, B_0, B_1 of $\mathcal{R}$.

(i) The set N_1 is independent of the choice of the sheets B_{-1}, B_0, B_1 if the assumptions made in Subsection 2.2 are satisfied and if in addition we have $N_1 \cap \Gamma_{-1} = \emptyset$.

(ii) If the assumptions formulated in part (i) are satisfied, then the set N_1 is the union of three analytic Jordan arcs Γ_{11}, Γ_{12}, Γ_{13}. Each of the three arcs Γ_{1j}, $j = 1, 2, 3$, connects the two branch points w_1 and w_4 in $\overline{\mathbb{C}}$, and at w_1 the arcs Γ_{11}, Γ_{12}, Γ_{13} have the tangential directions

$$\varphi_1 = \frac{65}{36}\pi, \qquad \varphi_2 = \frac{41}{36}\pi, \qquad \varphi_3 = \frac{17}{36}\pi, \tag{3.4}$$

respectively.

(iii) We have $\infty \in \Gamma_{13}$, and

$$w = \frac{1}{3}\log 2 + i\,\mathrm{Im}(w) + O\left(\frac{1}{|w|}\right) \quad \textit{as} \quad |w| \to \infty,\ w \in \Gamma_{13}. \tag{3.5}$$

(iv) For the intersection points of N_1 with $\mathbb{R}$ and $i\mathbb{R}$ we have the following numerical values

$$\Gamma_{11} \cap \mathbb{R} = \{0.59999\}, \quad \Gamma_{12} \cap \mathbb{R} = \{-0.3793\}, \quad \Gamma_{12} \cap i\,\mathbb{R} = \{-i\,0.621391,\ i\,0.621391\}. \tag{3.6}$$

Remark. The arcs Γ_{11}, Γ_{12}, and parts of the arc Γ_{13} are plotted in Figure 3.1.

3.3. Proof of Lemma 3.2. The proof of the lemma will be rather involved. However, a great part of the investigations will also be used in the subsequent proofs of lemmas and theorems from Section 2.

Let $\widetilde{\Gamma}_1, \widetilde{\Gamma}_{-1}$ be Jordan arcs that satisfy the assumptions made in Subsection 2.2. In addition to that we assume that $\widetilde{\Gamma}_{-1}$ is the reflection of $\widetilde{\Gamma}_1$ on the imaginary axis. The two arcs $\widetilde{\Gamma}_1, \widetilde{\Gamma}_{-1}$ determine the three sheets B_{-1}, B_0, B_1 of the Riemann surface $\mathcal{R}$, and consequently also the functions $h \circ \pi_1^{-1}$ and $h \circ \pi_0^{-1}$ used in definition (3.3) of N_1 are determined by the

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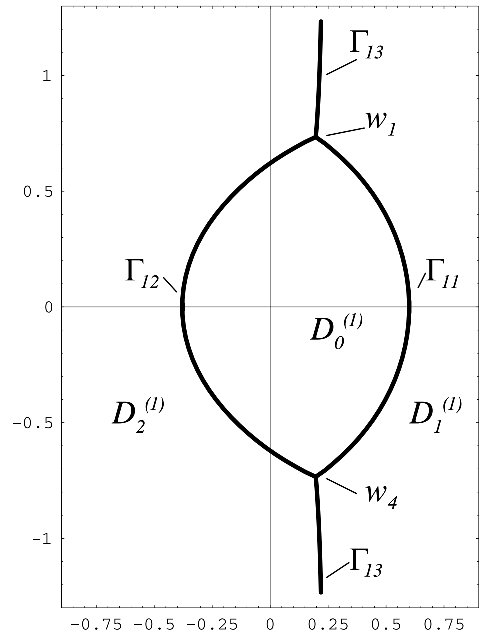

FIG. 3.1. *The arcs* Γ_{11},Γ_{12}, *and parts of the arc* Γ_{13}. *The three arcs form the set* N_1 *defined in (3.3). The three domains* $D_0^{(1)}$, $D_1^{(1)}$, $D_2^{(1)}$ *are the components of the set* $\mathbb{C}\backslash N_1$.

specific choice of the arcs $\widetilde{\Gamma}_1, \widetilde{\Gamma}_{-1}$. Since we have on $\widetilde{\Gamma}_1$ a change over between the two functions $h \circ \pi_1^{-1}$ and $h \circ \pi_0^{-1}$, the function

$$d(w) := h \circ \pi_1^{-1}(w) - h \circ \pi_0^{-1}(w) \tag{3.7}$$

changes sign when w crosses $\widetilde{\Gamma}_1$. From this we see that the definition of the set N_1 itself is independent of variations of the arc $\widetilde{\Gamma}_1$ if we have $\widetilde{\Gamma}_{-1} \cap N_1 = \emptyset$.

From Lemma 3.1 and the definitions of $h = \mathrm{Re}(u \circ \psi)$ and $h_j = h \circ \pi_j^{-1}$, $j = -1, 0, 1$, in Lemma 2.2 and Definition 2.3 we deduce that

$$\frac{\partial}{\partial x} h_j(w) = \mathrm{Re}\left[(u \circ \psi \circ \pi_j^{-1})'(w)\right] = 3\ \mathrm{Re}(v_j), \tag{3.8}$$

$$\frac{\partial}{\partial y} h_j(w) = -\,\mathrm{Im}\left[(u \circ \psi \circ \pi_j^{-1})'(w)\right] = -3\ \mathrm{Im}(v_j) \tag{3.9}$$

with $v_j := \psi \circ \pi_j^{-1}(w)$, $w = x + iy \in \overline{\mathbb{C}}$, $j = -1, 0, 1$. From the harmonicity of the function d it follows that the set N_1 consists of analytic arcs. From (3.3), (3.8-3.9), and (3.7) it further follows that

$$\left(\frac{\partial}{\partial x} d(w)\right)^2 + \left(\frac{\partial}{\partial y} d(w)\right)^2 = 9\ |v_1 - v_0|^2 . \tag{3.10}$$

It is a consequence of (3.10) that the arcs of N_1 can have no bifurcations in $\overline{\mathbb{C}} \setminus \{w_1, \ldots, w_4\}$ since otherwise we would have

$$\frac{\partial}{\partial x} d(w) = \frac{\partial}{\partial y} d(w) = 0, \qquad w = x + iy, \tag{3.11}$$

at such a point, which, however, is only possible at the branch points w_1 and w_4 because of (3.10).

It is immediate that $w_1,\, w_4 \in N_1$. In order to understand the structure of the set N_1 in a neighborhood of the branch point w_1, we consider the development of the function h in the local coordinate $\zeta = w_1 + \eta^2$. From Lemma 3.1 and the definitions in Subsection 2.2 we derive the development

$$u \circ \psi(\zeta) = u(v_1) + 3\, v_1 \eta^2 + \frac{2\sqrt{2}}{\sqrt{(\pi \circ \psi^{-1})''(v_1)}} \eta^3 + \dots \tag{3.12}$$

with $v_j := \psi(\zeta_1) = \sqrt[4]{-1/3}$. Evaluating $(\pi \circ \psi^{-1})''(v_1) = w''(v_1)$ then yields

$$h(\zeta) = h(\zeta_1) + 3\ \mathrm{Re}\left(v_1 \eta^2\right) + \mathrm{Re}\left(\sqrt[4]{\frac{(-1)^{1/4}}{3^2 - i\, 3^{3/2}}}\eta^3\right) + \dots \tag{3.13}$$

Taking into account the form of the two branches π_1^{-1} and π_0^{-1} in a neighborhood of the branch point w_1, we deduce from (3.7) and (3.13) that

$$d(w_1 + \eta^2) = 2\, |\eta|^3\, \mathrm{Re}\left(\sqrt[4]{\frac{(-1)^{1/4}}{3^2 - i\, 3^{3/2}}} e^{i\, 3\, \arg(\eta)/2}\right) + \mathrm{O}(|\eta|^4) \text{ as } |\eta| \to 0. \tag{3.14}$$

The condition $d(w_1 + \eta^2) = 0$ then implies that for $|\eta| \to 0$ we have

$$-\frac{15}{72}\pi + \frac{3}{2} \arg(\eta^2) \equiv \frac{1}{2}\pi + \mathrm{mod}(\pi), \tag{3.15}$$

and this implies that the set N_1 consists of three arcs in a neighborhood of w_1, which have the tangential directions given in (3.4) at w_1.

Using the mapping function (2.11), one can compare the behavior of the values $v_1 := \psi \circ \pi_1^{-1}(w)$ and $v_0 := \psi \circ \pi_0^{-1}(w)$ while w runs through $\mathbb{R}_+$. It then is possible to verify that

$$\mathrm{Re}(v_1(w) - v_0(w)) \begin{cases} > 0 & \text{for} \quad w > (\widetilde{\Gamma}_1 \cap \mathbb{R}) \\ < 0 & \text{for} \quad w < (\widetilde{\Gamma}_1 \cap \mathbb{R}) \end{cases}. \tag{3.16}$$

In a similar, but somewhat more involved way, one can show that

$$\mathrm{Im}(v_1(w) - v_0(w)) > 0 \quad \text{for all} \quad w \in i\,\mathbb{R}_+. \tag{3.17}$$

From (3.7), (3.8-3.9), (3.16), and (3.17) we deduce that

$$\frac{\partial}{\partial x} d(w) \begin{cases} < 0 & \text{for} \quad w > (\widetilde{\Gamma}_1 \cap \mathbb{R}) \\ > 0 & \text{for} \quad w < (\widetilde{\Gamma}_1 \cap \mathbb{R}) \end{cases} \tag{3.18}$$

and

$$\frac{\partial}{\partial y} d(w) > 0 \quad \text{for all} \quad w \in i\,\mathbb{R}_+, \tag{3.19}$$

i.e., the function d is strictly monotonic on $\mathbb{R}_+$, $i\,\mathbb{R}_+$, and $i\,\mathbb{R}_-$. On $\mathbb{R}_+$ at the intersection point of $\widetilde{\Gamma}_1$ and $\mathbb{R}$ we have a change over between the two branches h_1 and h_0, and therefore a sign change of d. In (3.16) and (3.18), the intersection point of $\widetilde{\Gamma}_1$ and $\mathbb{R}$ are denoted by $(\widetilde{\Gamma}_1 \cap \mathbb{R})$.

Let S_φ denote the ray $\{\, w = re^{i\varphi} \mid 0 \le r \le \infty \,\}$, $\varphi \in \mathbb{R}$. If we ignore for the moment the definition of the sheets B_{-1}, B_0, B_1 based on $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_{-1}$, and thereby also the specific definition of the functions $h_j = h \circ \pi_j^{-1}$, $j = 0, 1$, we can consider the function d by harmonic continuation along S_φ starting from ∞. We will denote this continuation by $\widetilde{d}$. From the discussion of the Riemann surface $\mathcal{R}$ in Subsection 2.2, we deduce that if $-\frac{5}{12}\pi < \varphi < \frac{5}{12}\pi$, then we have

$$\widetilde{d}(\infty) = d(\infty) = \infty \qquad \text{and} \qquad \widetilde{d}(0) = -d(0) = -\infty. \tag{3.20}$$

From (3.20) we conclude that each ray S_φ with $-\frac{5}{12}\pi < \varphi < \frac{5}{12}\pi$ has to cut the set N_1 in an odd number of points.

On the other hand we deduce from the monotonicity (3.19) on $i\,\mathbb{R}_+$, and $i\,\mathbb{R}_-$ that the set N_1 has at most one intersection with each half axis $i\,\mathbb{R}_+$, or $i\,\mathbb{R}_-$.

From the last two assertions it follows that the set N_1 contains a subarc Γ_1 that connects the two branch points w_1 and w_4, and this arc is contained in $\{\mathrm{Re}(w) > 0\}$. In the remainder of the proof we use the arc Γ_1 in place of $\widetilde{\Gamma}_1$, and as before we assume that Γ_{-1} is the reflection of Γ_1 on the imaginary axis. The sheets B_{-1}, B_0, B_1 and the functions $h_j = h \circ \pi_j^{-1}$, $j = -1, 0, 1$, are now assumed to be defined by the two arcs Γ_1 and Γ_{-1}. We collect some properties and consequences of the specific selection of Γ_1:

(a) Γ_1 is an analytic Jordan arc connecting the two points w_1 and w_4 in $\{\, \mathrm{Re}(w) > 0 \,\}$, and it satisfies the assumptions made in Subsection 2.2.

(b) The function $h_1 = h \circ \pi_1^{-1}$ is continuous in $\mathbb{C}$. Indeed, since $d(w) = 0$ for all $w \in \Gamma_1$, it follows that $h_1(w_+) = h_1(w_-)$ if w_+ and w_- denote the two sides of Γ_1 at a point $w \in \Gamma_1$.

(c) The function h_1 is harmonic in $\mathbb{C} \backslash \Gamma_1$ and has the representation

$$h_1(w) - 3\ \mathrm{Re}(w) \; = \; -\int \log \frac{1}{|w - x|} d\nu_1(x) \tag{3.21}$$

with a measure ν_1 on Γ_1 satisfying

$$\|\nu_1\| = 1 \tag{3.22}$$

and

$$d\nu_1(w) \; := \; \frac{3}{2\pi} \, |v_+(w) - v_-(w)| \, ds_w, \qquad w \in \Gamma_1, \tag{3.23}$$

where $v_+(w)$ and $v_-(w)$ denote the two points $\psi \circ \pi_1^{-1}(w_+) \; = \; \psi \circ \pi_1^{-1}(w)$ and $\psi \circ \pi_1^{-1}(w_-) = \psi \circ \pi_0^{-1}(w)$, respectively.

Indeed, it follows from the definition of h in Lemma 2.2 that the function h_1 is harmonic in $\mathbb{C} \backslash \Gamma_1$. From the development (2.16) in Lemma 2.2 we deduce that

$$h_1(w) - 3\ \mathrm{Re}(w) \; = \; \log|w| + \mathrm{O}(1/\,|w|) \quad \text{as} \quad |w| \to \infty, \tag{3.24}$$

and with the help of the Green formula (cf. [20]) it then follows that

$$h_1(w) - 3\ \mathrm{Re}(w) \; = \; -\frac{1}{2\pi} \int_{\Gamma_1} \log \frac{1}{|w - x|} \left(\frac{\partial}{\partial n_+} h_1(x) + \frac{\partial}{\partial n_-} h_1(x) \right) ds_x \tag{3.25}$$

for $w \in \mathbb{C} \backslash \Gamma_1$, where ds_x denotes the line element on Γ_1, and $\partial/\partial n_\pm$ the normal derivatives to both sides of Γ_1 at the point $x \in \Gamma_1$. Let $t = t_w, n_+ = n_{w+}, n_- = n_{w-} \in \partial\mathbb{D}$ denote the tangent and the two normal vectors on Γ_1 at an inner point $w \in \Gamma_1$. From the definition of Γ_1 we know that

$$\frac{\partial}{\partial t_w} d(w) = \frac{\partial}{\partial t_w} \left(h_1(w_+) - h_1(w_-) \right) = 0 \quad \text{for} \quad w \in \Gamma_1, \tag{3.26}$$

with w_+ and w_- denoting the two sides of Γ_1 at the point $w \in \Gamma_1$, as in assertion (b). From (3.26) and Lemma 3.1 we derive that

$$\begin{aligned} \frac{\partial}{\partial n_+} h_1(w) + \frac{\partial}{\partial n_-} h_1(w) &= \frac{\partial}{\partial n_+} \left(h_1(w_+) - h_1(w_-) \right) \\ &= 3 \left| v_{1+}(w) - v_{1-}(w) \right|, \end{aligned} \tag{3.27}$$

where $v_{1+}(w) = \psi \circ \pi_1^{-1}(w_+) = \psi \circ \pi_1^{-1}(w)$ and $v_{1+}(w) = \psi \circ \pi_1^{-1}(w_+) = \psi \circ \pi_0^{-1}(w)$. In order to justify the last equality in (3.27), we remark that from (3.26) and from the fact that the tangent t_w and the normal vectors $n_{w\pm}$ are orthogonal, it follows that $v_{1+}(w) - v_{1-}(w) \in \mathbb{R}$ for all $w \in \Gamma_1$. Since for all inner points $w \in \mathrm{Int}(\Gamma_1)$ we have $v_{1+}(w) - v_{1-}(w) \neq 0$, it follows that the function on the left-hand side of (3.27) can have no sign change on Γ_1. From (3.24) we conclude that the functions in (3.27) are non-negative for all $w \in \Gamma_1$. With (3.27) we have proved (3.23). Assertion (3.22) follows again from (3.24).

Next, we look at the global behavior of the function d. It follows from the behavior of the two functions h_1 and h_0 at infinity, which are stated in Lemma 2.2, that for $c > 0$ sufficiently large we have $d(w) > 0$ for $\mathrm{Re}(w) > c$ and $d(w) < 0$ for $\mathrm{Re}(w) < -c$. At the origin we have $d(0) = \infty$. After (3.10) it has been concluded that the set N_1 has no bifurcations in $\overline{\mathbb{C}} \setminus \{w_1, w_4\}$. From (3.16) and symmetry of the arc Γ_1 and the functions h_1, h_0 with respect to $\mathbb{R}$, we know that the set N_1 has exactly three arcs ending in each of the two branch points w_1 and w_4. Consequently, the set $\mathbb{C} \setminus N_1$ can have at most three components, and our earlier considerations show that it has exactly three components. In Figure 3.1 these three components are denoted by $D_j^{(1)}$, $j = 0, 1, 2$. The two domains $D_0^{(1)}$ and $D_1^{(1)}$ are separated by the Jordan arc $\Gamma_{11} = \Gamma_1$, the two domains $D_0^{(1)}$ and $D_2^{(1)}$ by the Jordan arc Γ_{12}, and the two domains $D_1^{(1)}$ and $D_2^{(1)}$ by the Jordan arc Γ_{13}. In the interest of a unified notation we have renamed Γ_1 into Γ_{11}. Numerical values for the intersection points of the set N_1 with $\mathbb{R}$ and $i\mathbb{R}$ are given in (3.6).

For later use, we consider the representation of the function

$$\widetilde{d}(w) := \begin{cases} d(w) & \text{for } w \in D_0^{(1)} \\ 0 & \text{elsewhere} \end{cases} . \tag{3.28}$$

as a logarithmic potential. We have

$$\widetilde{d}(w) = -\int_{\Gamma_{11} \cup \Gamma_{12}} \log \frac{|w - x|}{|w|} d(\nu_1 + \widetilde{\psi}_1)(x) \tag{3.29}$$

with ν_1 the measure on Γ_{11} given by (3.23), and the measure $\widetilde{\psi}_1$ is defined by

$$d\widetilde{\psi}_1(w) := \frac{3}{2\pi} \left| v_1(w) - v_0(w) \right| ds_w, \qquad w \in \Gamma_{12}. \tag{3.30}$$

We have

$$\left\| \widetilde{\psi}_1 \right\| = 2. \tag{3.31}$$

Indeed, in the same way as done in (3.21) it follows from Green's formula and from the considerations that have led to the identities in (3.27) that

$$\widetilde{d}(w) + 3\,\log|w| \;=\; -\frac{1}{2\pi}\oint_{\Gamma_{11}\cup\Gamma_{12}} \log\frac{1}{|w-x|}\,\frac{\partial}{\partial n_+}d(w)ds_x, \tag{3.32}$$

where $\partial/\partial n_+$ denotes the inwardly pointing normal derivative on $\Gamma_{11}\cup\Gamma_{12}$. As in the derivation (3.26) and (3.27) it follows that

$$\frac{\partial}{\partial n_+}d(w) = 3\;|v_1(w) - v_0(w)|\,ds_w, \quad \text{for} \quad w\in\Gamma_{11}\cap\Gamma_{12} \tag{3.33}$$

with $v_1(w) = \psi\circ\pi_1^{-1}(w)$ and $v_0(w) = \psi\circ\pi_0^{-1}(w)$. The definition of ν_1 in (3.23) shows that the measure defined by (3.32) on Γ_{11} is identical with the measure defined in (3.23). Identity (3.30) follows from (3.32) and (3.33), and (3.31) is a consequence of the fact that d has a logarithmic singularity at the origin with residue 3.

It remains to show that the asymptotic relation (3.5) holds true. From (2.16) and (2.18) in Lemma 2.2 together with the symmetry of the functions h_1 and h_{-1} with respect to $\mathbb{R}$, we deduce that

$$h_1(w) \;=\; 3\;\mathrm{Re}(w) + \log|w| + \mathrm{O}(\frac{1}{|w|}) \quad \text{as} \quad |w|\to\infty. \tag{3.34}$$

If we put together the development of function (3.15) at $v = 0$ and the development of the function ψ, which has been introduced in Subsection 2.2 and has to be developed at $\infty^{(0)}$, then we arrive at the development

$$h_0(w) \;= \log\,2 + \log|w| + \mathrm{O}(\frac{1}{|w|}) \quad \text{as} \quad |w|\to\infty. \tag{3.35}$$

Hence, we have

$$d(w) \;=\; 3\;\mathrm{Re}(w) - \log\,2 + \mathrm{O}(\frac{1}{|w|}) \quad \text{as} \quad |w|\to\infty. \tag{3.36}$$

From (3.36) and $d(w) = 0$ for $w\in\Gamma_{13}$, the asymptotic relation (3.5) then follows. $\square$

3.4. Proof of Lemma 2.4. It is an immediate consequence of the maximum principle for harmonic functions that the characterization of the arcs Γ_1, Γ_{-1} and of the sets K_0, K_∞, given in the lemma, determines these objects uniquely. Thus, only the existence of the objects has to be proved.

The existence of the two arcs Γ_1, Γ_{-1} follows for the arc $\Gamma_1 = \Gamma_{11}$ from the properties (a) and (b) that have been established in the proof of Lemma 3.2 after (3.20); for the arc Γ_{-1} it follows from the symmetry between the two arcs Γ_1 and Γ_{-1}.

The proof of existence of the set $K_0\subseteq\mathbb{C}$ is based on results established in Lemma 3.2 and its proof. Let the set K_0 be defined by the following three properties: (a) The set K_0 is symmetric with respect to $i\mathbb{R}$, (b) we have $K_0\cap\{\mathrm{Re}(w)>0\} = \Gamma_{12}\cap\{\mathrm{Re}(w)>0\}$ with the arc Γ_{11} introduced in Lemma 3.2, and (c) $K_0\cap i\mathbb{R}$ is the interval connecting the two intersection points of Γ_{12} with $i\mathbb{R}$. The continuation h_0 of the function element $\widetilde{h}_0$ at infinity is given by $h\circ\pi_0^{-1}$ in $\overline{D_1^{(1)}}$ and by $h\circ\pi_1^{-1}$ in $\overline{D_0^{(1)}\cup D_2^{(1)}}\cap\{\mathrm{Re}(w)>0\}$, where $D_j^{(1)}$, $j=0,1,2$, denote the three components of $\mathbb{C}\backslash N_1$ as indicated in Figure 3.1. This establishes the definition of h_0 in the half-plane $\{\mathrm{Re}(w)\geq 0\}$. In the other half-plane $\{\mathrm{Re}(w)<0\}$ the continuation h_0 of the function element $\widetilde{h}_0$ is determined by the symmetry with respect to the imaginary axis. It follows from the properties of $h \circ \pi_1^{-1}$ and $h \circ \pi_0^{-1}$ studied in the proof of Lemma 3.2, that the continuation h_0 is continuous in $\mathbb{C}$. The other requirements made in part (ii) of the lemma are also satisfied. The assertions of part (ii) including (2.24) then follow from Lemma 3.2.

The existence of the set $K_\infty \subseteq \overline{\mathbb{C}}$ in part (iii) of the lemma, can be shown in a way very similar to that used for K_0. But now we start from the function element $\widetilde{h}_\infty$ defined at the origin. Instead of the arc Γ_{12} we use the arc Γ_{13} introduced in Lemma 3.2. $\square$

3.5. Proof of Lemma 2.6. The existence of representation (2.27) for h_1 has already been worked out in detail in the proof of Lemma 3.2; the results are contained in (3.21) and (3.23). Representation (2.25) for h_{-1} then follows from the symmetry of the two functions h_1 and h_{-1} with respect to the imaginary axis.

The existence of representation (2.26) for h_0 can be deduced in a way which is quite analogous to the derivation of the representations (3.21) and (3.23). In the new situation one has to use the explicit definition of the set K_0 given in the proof of Lemma 2.4. We remark that for instance we have $\nu_0\big|_{\{\mathrm{Re}(w)>0\}} = \widetilde{\psi}_1\big|_{\{\mathrm{Re}(w)>0\}}$ with $\widetilde{\psi}_1$ defined by (3.30). The constant term in (2.26) follows from development (3.35). $\square$

3.6. Proof of Lemma 2.9. As in Lemma 2.4, and also here, the uniqueness of the arcs $\Gamma_{-1,2}$, Γ_{12}, the set K_1, and the harmonic continuations g_{-1}, g_0, g_1 follow from the requirements formulated in the lemma and the maximum principle for harmonic functions. Thus, we have only to prove the existence of these objects.

Let Γ_{12} be the arc Γ_{12} introduced in Lemma 3.2. Let further the harmonic continuation g_1 of a function element $\widetilde{h}_1$ at infinity be defined by $h \circ \pi_1^{-1}$ on $\overline{D_0^{(1)} \cup D_2^{(1)}}$ and by $h \circ \pi_0^{-1}$ in $D_1^{(1)}$ with $D_j^{(1)}$, $j = 0, 1, 2$, being the three components of the set $\mathbb{C} \setminus N_1$ as indicated in Figure 3.1. We assume that the sheets B_{-1}, B_0, B_1 of $\mathcal{R}$ are those fixed in the second part of the proof of Lemma 3.2 after (3.20). These sheets determine the branches π_1^{-1} and π_0^{-1}. The function g_1 defined in this way satisfies all conditions formulated in (i), (ii), and (iii) of the lemma.

Let the harmonic continuation g_{-1} of a function element $\widetilde{h}_{-1}$ at infinity be defined as the symmetric counterpart of the function g_1 under reflection on the imaginary axis. The arc $\Gamma_{-1,2}$ also is the image of Γ_{12} under reflection on the imaginary axis. It is immediate that these definitions satisfy the requirements of the lemma.

Let the harmonic continuation g_0 of a function element $\widetilde{h}_0$ at infinity be defined as $h \circ \pi_0^{-1}$ in the domain $\overline{\mathbb{C}} \setminus K_1$ with K_1 defined as $K_1 := \Gamma_{-1} \cap \Gamma_1$. From Lemma 3.2 and the definition of N_1 in (3.3) we learn that g_0 is continuous in a neighborhood of $\Gamma_1 = \Gamma_{11}$. From the symmetry of the function h_0 with respect to the imaginary axis, it then follows that continuity also holds in a neighborhood of Γ_{-1}. Thus, also g_0 and K_1 satisfy the requirements of the lemma. $\square$

3.7. Proof of Lemma 2.10. Part (i) of the lemma has already been verified in the proof of Lemma 2.9. It has also been shown in the proof of Lemma 2.9 that the arc $\Gamma_{-1,2}$ is the image of Γ_{12} under reflection on the imaginary axis. The first assertion in part (ii) of the lemma follows from Lemma 3.2 for Γ_{12}. For the arc $\Gamma_{-1,2}$ it then follows as a consequence of the symmetry between the two arcs Γ_{12} and $\Gamma_{-1,2}$. The intersections of the arcs Γ_{12} and $\Gamma_{-1,2}$ with the set K_0, stated in the lemma, follow from the detailed description of the set K_0 in the proof of Lemma 2.4.

Part (iii) of the lemma follows from Lemma 3.2 for the arc Γ_{12}, and for the arc $\Gamma_{-1,2}$ it then follows as a consequence of the symmetry between the two arcs. $\square$

3.8. Proof of Lemma 2.11. We start with the proof of representation (2.36). Since g_1 is the harmonic continuation of the function element $\widetilde{h}_1$ at infinity, we know from (2.18) and (2.16) in Lemma 2.2 that g_1 has the development

$$g_1(w) = 3\ \mathrm{Re}(w) + \log|w| + \mathrm{O}(1/|w|) \quad \text{as} \quad |w| \to \infty, \tag{3.37}$$

and from (2.19) that

$$g_1(w) = 3\ \log|w| + \mathrm{O}(1) \qquad \text{as} \quad |w| \to 0. \tag{3.38}$$

By Green's formula we deduce, from the properties of g_1 and from (3.37) and (3.38) in a way very analogous to the derivation of the representations (3.21) and (3.23), that

$$g_1(w) - 3\ \mathrm{Re}(w) = 3\ \log|w| - \frac{1}{2\pi}\int_{\Gamma_{12}} \log\frac{1}{|w-x|}\left(\frac{\partial}{\partial n_+}h_0(x) + \frac{\partial}{\partial n_-}h_0(x)\right) ds_x \tag{3.39}$$

for $w \in \mathbb{C}\setminus(\Gamma_{12}\cup\{0\})$. Representation (2.36) then follows from (3.39) with the definition

$$\begin{aligned} d\psi_1(w) &:= \frac{1}{4\pi}\left(\frac{\partial}{\partial n_+}h_0(w) + \frac{\partial}{\partial n_-}h_0(w)\right) ds_w, \\ &= \frac{3}{4\pi}\,|v_1(w) - v_0(w)|\, ds_w, \qquad w \in \Gamma_{12}, \end{aligned} \tag{3.40}$$

where $v_1(w) = \psi\circ\pi_1^{-1}(w)$ and $v_0(w) = \psi\circ\pi_0^{-1}(w)$. The last equality in (3.40) is a consequence of Lemma 3.1. The argument is the same as that given after (3.27) in the proof of Lemma 3.2. A comparison with (3.29), (3.30), and (3.31) shows that $\widetilde{\psi}_1 = 2\,\psi_1$, which implies that ψ_1 in (2.36) is a probability measure.

Representation (2.34) for the function g_{-1} follows from (2.36), because of the symmetry between the two functions g_1 and g_{-1}.

Representation (2.35) follows from (3.35), the representations (2.25) and (2.27) given in Lemma 2.6, and the fact that the difference $h_1 + g_0$ is harmonic in a neighborhood of Γ_1, and $h_{-1} + g_0$ is harmonic in a neighborhood of Γ_{-1}. □

3.9. Proof of Theorem 2.14. The parts (i) and (ii) of the theorem are rather immediate consequences of the definition of the mapping (2.11) and its inverse function ψ together with the definition of the function h in Lemma 2.2. The two parts (iii) and (iv) of the theorem are direct translations of the method of analytic continuation in a given domain. □

3.10. Proof of Theorem 2.16. Part (i) of the theorem has been proved in Lemma 3.2. For the proof of part (ii) we first consider an expression for the directional derivative $\partial/\partial t_w$ at a point w with $t_w \in \partial\mathbb{D}$ the given direction of the derivative. Let $d = h\circ\pi_1^{-1} - h\circ\pi_0^{-1}$ be defined as in (3.7). From (3.8), (3.9), and Lemma 3.1 we deduce that

$$\begin{aligned} \frac{\partial}{\partial t_w}d(w) &= \frac{\partial}{\partial t_w}\left(\mathrm{Re}\left[(u\circ\psi\circ\pi_1^{-1})(w)\right] - \mathrm{Re}\left[(u\circ\psi\circ\pi_0^{-1})(w)\right]\right) \\ &= \mathrm{Re}\left[t_w(u\circ\psi\circ\pi_1^{-1})'(w) - t_w(u\circ\psi\circ\pi_2^{-1})'(w)\right] \\ &= 3\ \mathrm{Re}\left[t_w(v_1(w) - v_0(w))\right], \qquad w \in N_1\setminus\{w_1, w_4\}. \end{aligned} \tag{3.41}$$

If $t_w \in \partial\mathbb{D}$ is the tangential direction at a point w of one of the three open arcs of $N_1 \setminus \{w_1, w_4\}$, then from (3.3) it is immediate that $(\partial/\partial t_w)d(w) = 0$ for all $w \in N_1\setminus\{w_1, w_4\}$. We therefore deduce from (3.41) that

$$t_w = \pm i\,\frac{\overline{v_1(w) - v_0(w)}}{|v_1(w) - v_0(w)|} \qquad \text{for} \quad w \in N_1\setminus\{w_1, w_4\}, \tag{3.42}$$

where the points v_1 and v_0 are given by $v_1(w) := \psi\circ\pi_1^{-1}(w)$ and $v_0(w) := \psi\circ\pi_0^{-1}(w)$, $w \in N_1 \setminus \{w_1, w_4\}$. These definitions translate into the notation $v_{1+}(w)$ and $v_{1-}(w)$, $w \in \Gamma_{1j}$, $j = 1, 2, 3$, which has been used in formula (2.47) of the theorem.

The parts (iii) and (iv) of the theorem are covered by the proof of Lemma 2.4. □

3.11. Proof of Theorem 2.17. Representations, for all measures that appear in the conclusions (2.56-2.59) of the theorem, can be derived in the same way as the formulae (3.23) and (3.30) in the proof of the Lemma 3.2. As in the proof of Theorem 2.16, so also here, the notations v_{j+} and v_{j-}, $j = 1, 2, 3$, translate directly to the notations used in the proof of Lemma 3.2. □

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